PROPERTIES OF GENERALIZED DERANGEMENT GRAPHS

HANNAH JACKSON, KATHRYN NYMAN, AND LES REID

Abstract. A permutation \( \sigma \in S_n \) is a \( k \)-derangement if for any subset \( X = \{a_1, \ldots, a_k\} \subseteq [n], \{\sigma(a_1), \ldots, \sigma(a_k)\} \neq X \). One can form the \( k \)-derangement graph on the set of permutations of \( S_n \) by connecting two permutations \( \sigma \) and \( \tau \) if \( \sigma \tau^{-1} \) is a \( k \)-derangement. We characterize when such a graph is connected or Eulerian. For \( n \) an odd prime power, we determine the independence, clique and chromatic number of the 2-derangement graph.

1. Introduction

Permutations which leave no element fixed, known as derangements, were first considered by Pierre Raymond de Montmort in 1708 and have been extensively studied since. A derangement graph is a graph whose vertices are the elements of the symmetric group \( S_n \) and whose edges connect two permutations \( \sigma \) and \( \tau \) that differ by a derangement. Derangement graphs have been shown to be connected (for \( n > 3 \)), Hamiltonian, and their independence number, clique number, and chromatic number have been calculated [3].

The concept of a derangement can be generalized to a \( k \)-derangement, a permutation in \( S_n \) such that the induced permutation on the set of all unordered \( k \)-tuples leaves no \( k \)-tuple fixed. A \( k \)-derangement graph is defined in an analogous manner to a derangement graph. In this paper, we investigate some of the graph-theoretical properties of \( k \)-derangement graphs.

2. Preliminaries

Let \( S_n \) be the group of permutations on the set \( [n] = \{1, 2, \ldots, n\} \), and denote by \( [n]^{(k)} \) the set of unordered \( k \)-tuples with entries from \( [n] \). Note that a permutation \( \sigma \in S_n \) induces a permutation \( \sigma^{(k)} \) of unordered \( k \)-tuples by \( \sigma^{(k)}(\{a_1, \ldots, a_k\}) = \{\sigma(a_1), \ldots, \sigma(a_k)\} \). For example, with \( n = 4 \), \( k = 2 \), and \( \sigma = (1234) \) in cycle notation, we have

\[
\begin{align*}
(1234)_{(2)}(\{1, 2\}) &= \{(1234)(1), (1234)(2)\} = \{2, 3\} \\
(1234)_{(2)}(\{1, 3\}) &= \{(1234)(1), (1234)(3)\} = \{2, 4\} \\
(1234)_{(2)}(\{1, 4\}) &= \{(1234)(1), (1234)(4)\} = \{2, 1\} = \{1, 2\} \\
(1234)_{(2)}(\{2, 3\}) &= \{(1234)(2), (1234)(3)\} = \{3, 4\} \\
(1234)_{(2)}(\{2, 4\}) &= \{(1234)(2), (1234)(4)\} = \{3, 1\} = \{1, 3\}
\end{align*}
\]

Date: July 29, 2011.

2000 Mathematics Subject Classification. 05C69, 05A05, 05C45.

Key words and phrases. Derangements, Eulerian, Chromatic Number, Maximal Clique, Cayley Graph, Independent Set.
Indeed, if \( \Gamma \) by \( \Gamma(\sigma) \) in \( S \) is in \( S_k \) of cycles whose lengths partition \( k \) tend this concept, we say that a permutation \( \sigma \in S_n \) is a \( k \)-derangement if \( \sigma(x) \neq x \) for all \( x \in [n] \). In other words, a \( k \)-derangement in \( S_n \) is a permutation (of \([n]\)) which induces a permutation (of \([n]\)) which leaves no \( k \)-tuple fixed. The set of \( k \)-derangements in \( S_n \) is denoted \( D_{k,n} \), and the number of \( k \)-derangements in \( S_n \) is denoted \( D_k(n) \). The example above shows that \((1234)\) is in \( D_2 \). Specifically, \( D_2 = \{ (1234), (1243), (1324), (1342), (1423), (1432), (123)(4), (124)(3)(132)(4), (134)(2), (142)(3), (143)(2), (234)(1), (243)(1) \} \), and thus \( D_2(4) = 14 \). Note that \( D_n = D_{1,n} \), and \( D_{1}(n) \) is the ordinary derangement number.

The cycle structure of a permutation \( \sigma \), denoted \( C_\sigma \), is the multiset of the lengths of the cycles in its cycle decomposition (e.g., \( C_{(12)(3)(45)} = \{2, 2, 1\} \)). Note that the cycle structure of \( \sigma \in S_n \) is a partition of \( n \). Given a partition \( r \vdash n \), let \( P_r \) be the set of all permutations in \( S_n \) whose cycle structure is \( r \). For example, \( P_{[2,1,1]} = \{ (12), (13), (14), (23), (24), (34) \} \).

We first note that if the cycle structure of a permutation \( \sigma \) contains a multiset which partitions \( k \), then \( \sigma \) is not a \( k \)-derangement. For example, \((12)(3)(4)\) will be a 3-derangement in \( S_4 \), but \((12)(3)(4)\) will not be, because \( \{2,1,1\} \subseteq C_{(12)(3)(4)} = \{2, 1, 1\} \) is a partition of 3.

Indeed, if \( q, r, \ldots, s \) is a partition of \( k \), and \( (a_1 \ldots a_q)(b_1 \ldots b_r) \ldots (c_1 \ldots c_s) \) are cycles of \( \sigma \), then for \( x = \{a_1, \ldots, a_q, b_1, \ldots, b_r, c_1, \ldots, c_s\} \), \( \sigma(k)(x) = x \). Conversely, if \( \sigma \) has no set of cycles whose lengths partition \( k \), then given any \( x \in [n]^{(k)} \), there is a cycle in \( \sigma \) which contains at least one element in \( x \) and contains some element not in \( x \). Hence \( \sigma \) sends an element in \( x \) to an element not in \( x \) and so \( \sigma(k)(x) \neq x \).

Thus we observe that the cycle structure of a permutation determines whether or not it is a \( k \)-derangement, and we have the following.

**Proposition 1.** A permutation \( \sigma \in S_n \) is a \( k \)-derangement if and only if the cycle decomposition of \( \sigma \) does not contain a set of cycles whose lengths partition \( k \).

Let \( CD_{k,n} \) be the set of cycle structures corresponding to \( k \)-derangements in \( S_n \) [e.g., \( CD_{2,4} = \{ \{4\}, \{3, 1\} \} \)]. Note that \( D_{k,n} = D_{(n-k),n} \). This follows from the fact that if a cycle structure \( C_\sigma \) in \( CD_{k,n} \), then \( C_\sigma \) is in \( CD_{(n-k),n} \) as well.

Let \( G \) be a group, and let \( S \subseteq G \) such that if \( s \) is in \( S \), then \( s^{-1} \) is in \( S \). The Cayley graph \( \Gamma(G, S) \) is the graph whose vertices are the elements of \( G \) such that an edge connects two vertices \( u, v \in G \) if \( su = v \) for some \( s \in S \). A \( k \)-derangement graph is a Cayley graph defined by \( \Gamma_{k,n} := \Gamma(S_n, D_{k,n}) \). (Note that \( D_{k,n} \) is symmetric, as the inverse of a \( k \)-derangement is a \( k \)-derangement, and thus satisfies the requirements for a Cayley graph.) It is worth noting that \( \Gamma_{k,n} \) is, by construction, \( D_k(n) \)-regular, and that since \( D_{k,n} = D_{(n-k),n} \), \( \Gamma_{k,n} = \Gamma_{(n-k),n} \).

Figure 1 illustrates the 2-derangement graph on 6 vertices, \( \Gamma_{2,3} \).

It is possible to consider \( k \)-derangements in \( S_n \) for any positive \( k \) and \( n \). However, if \( k = n \), there will be no \( k \)-derangements in \( S_n \), since every partition in \( S_n \) will have a cycle structure such that the cycle lengths partition \( k \). As such, \( \Gamma_{k,n} \) will be the empty graph on \( n \) vertices. If \( k > n \), then every permutation in \( S_n \) is a \( k \)-derangement vacuously, and thus
Γ_{k,n} will be the complete graph on |S_n| vertices. As neither of these cases is particularly interesting, henceforth we will only consider k-derangements where k < n.

3. Properties of derangement graphs

Figure 1 shows that Γ_{2,3} is not a connected graph, and since Γ_{2,3} = Γ_{1,3}, we see that Γ_{k,3} is disconnected, for all k < n. But this is an exception rather than the rule, as the following theorem demonstrates.

Theorem 2. For n > 3 and k < n, Γ_{k,n} is connected.

Proof. Every permutation in S_n can be written as the product of adjacent transpositions (h (h + 1)). These, in turn, can be expressed as the product of two k-derangements, so long as n > 3, as we will demonstrate. As a result, for n > 3, the elements of D_{k,n} generate S_n, which means that every vertex of Γ_{k,n} can be reached by a path from the identity.

We show that the permutation (1 2) can be written as the product of two k-derangements and then note that since it is the form and not the individual labels that are important, any adjacent transposition can be written as the product of two k-derangements. We consider two cases, the case where k = 1, and the case where k ≥ 2.

Case 1: If k = 1, then (1 2) = (1 2 ... n)^2 · (n (n - 1) ... 1)^2(1 2). We claim that (1 2 ... n)^2 and (n (n - 1) ... 1)^2(1 2) are each 1-derangements in S_n for all n > 3. If n is even, then (1 2 ... n)^2 = (1 3 ... (n - 3) (n - 1))(2 4 ... (n - 2) n), which is a 1-derangement in S_n, for all n. Additionally, (n (n - 1) ... 1)^2(1 2) = (1 n (n - 2) (n - 4) ... 2 (n - 1) (n - 3) ... 3), which is also a 1-derangement in S_n, for any n.

On the other hand, if n is odd, then (1 2 ... n)^2 = (1 3 ... (n - 2) n 2 4 ... (n - 3) (n - 1)), which is a 1-derangement in S_n for all n. And (n (n - 1) ... 1)^2(1 2) = (n (n - 2) (n - 4) ... 3 1 (n - 1) (n - 3) ... 4 2)(1 2) = (1 n (n - 2) (n - 4) ... 3)(2 (n - 1) (n - 3) ... 4), which is a 1-derangement in S_n so long as n > 3. (If n = 3, (312)(12) = (13)(2), which is not a 1-derangement.)
Thus for \( n > 3 \), we have shown that \((1 2)\) can be written as the product of two 1-derangements, and, by extension, every adjacent transposition can be written as the product of two 1-derangements.

Case 2: For \( k \geq 2 \), \((1 2) = (1 2 \ldots n)^{-1}(1 3 4 \ldots n)\). We know \((1 2 \ldots n)^{-1}\) is a \(k\)-derangement for all \( k \) since the inverse of a \(k\)-derangement is a \(k\)-derangement. And, by the cycle structure, we see that \((1 3 4 \ldots n) = (1 3 4 \ldots n)(2)\) is a \(k\)-derangement for all \( k \), except \( k = 1 \) and \( k = (n - 1)\). (However, since \( \Gamma_{1,n} = \Gamma_{(n-1),n} \), Case 1 addresses \((n - 1)\)-derangements as well as 1-derangements).

So we have shown that for \( k \geq 2 \), \((1 2)\) can be written as the product of two \(k\)-derangements, and again, by extension, we can write any adjacent transposition as the product of two \(k\)-derangements. Thus every vertex is connected by a path to the identity, and \( \Gamma_{k,n} \) is connected.

□

It is worth noting that Theorem 1 holds for \( n = 2 \) as well. Since we are only interested in \(k\)-derangements in \( S_n \) such that \( k < n \), when \( n = 2 \), \( k \) must equal 1, and so \( \Gamma_{1,2} \) is the connected graph on two vertices.

Next, we give a characterization in terms of \( n \) and \( k \) for when a derangement graph is Eulerian. We will require the following result.

**Lemma 3.** If a cycle structure includes a cycle of length greater than 2, then there are an even number of permutations with that cycle structure.

**Proof.** Consider \( P_r \), the set of permutations with a given cycle structure, \( r \). We can pair each \( \sigma \in P_r \) with its inverse \( \sigma^{-1} \in P_r \), and so long as \( \sigma \neq \sigma^{-1} \) for any \( \sigma \in P_r \), \(|P_r|\) will be even. Suppose there exists a \( \sigma \in P_r \) such that \( \sigma = \sigma^{-1} \). Then \( \sigma^2 = e \), and so the order of \( \sigma \) is at most 2. The order of a permutation is the least common multiple of the orders of the elements of its cycle structure, so \( \sigma \) must not include a cycle of length greater than 2. This is a contradiction; thus \(|P_r|\) is even. □

**Theorem 4.** For \( n > 3 \) and \( k < n \), \( \Gamma_{k,n} \) is Eulerian if and only if \( k \) is even or \( k \) and \( n \) are both odd.

**Proof.** A graph is Eulerian if and only if it is connected and each vertex has an even degree. In light of Theorem 2 and the previously noted fact that \( \Gamma_{k,n} \) is \( D_k(n) \)-regular, in order to ascertain if \( \Gamma_{k,n} \) is Eulerian, we must determine whether \( D_k(n) \) is even or odd.

If \( k \) is even, we claim that \( D_k(n) \) is the sum of even numbers. Any cycle structure composed entirely of 2- or 1-cycles will partition an even \( k \), and thus any permutation which is in \( D_{k,n} \) for an even \( k \) will contain a cycle of length 3 or greater in its cycle decomposition. Now, \( D_{k,n} = P_{r_1} \cup P_{r_2} \cdots \cup P_{r_m} \) such that no \( r_i \) partitions \( k \), and by Lemma 3, \(|P_{r_i}|\) is even for all \( i \in \{1, \ldots, m\} \). Thus, when \( k \) is even, \( D_k(n) \) is even.

If \( k \) and \( n \) are both odd, again we see that every permutation in \( D_{k,n} \) will contain a cycle of length 3 or greater in its cycle decomposition, since an odd \( k \) can be partitioned by a set of cycles of lengths 1 or 2 if there is at least one 1-cycle. Furthermore, since \( n \) is
odd, there are no permutations whose cycle structure is composed only of length-2 cycles. Thus, \( D_k(n) \) is even.

Finally, we show that if \( k \) is odd and \( n \) is even, then \( \Gamma_{k,n} \) is not Eulerian. In this case, \( P_{(2,2,\ldots,2)} \) is in \( CD_{k,n} \). By choosing pairs of elements for the cycles and dividing by the number of ways to order the cycles, we see that the number of permutations in \( P_{(2,2,\ldots,2)} \) is given by

\[
\frac{n(n-1)(n-2)\cdots(3)(2)(1)}{(2\cdot\frac{n}{2})(2\cdot(\frac{n}{2}-1))\cdots(6)(4)(2)} = \frac{n(n-1)(n-2)\cdots(3)(2)(1)}{n(n-2)\cdots(6)(4)(2)} = (n-1)(n-3)\cdots(5)(3)(1).
\]

Since \( n \) is even, the product \((n-1)(n-3)\cdots(5)(3)(1)\) is odd. Every other \( k \)-derangement in \( S_n \) will contain a cycle with length greater than 2, since any combination of 1-cycles or 1- and 2-cycles will partition \( k \). So \( D_k(n) \) is the sum of one odd number and even numbers, and so is odd.

\[\square\]

4. Chromatic, independence and clique numbers for \( k = 2 \) and \( n \) an odd prime power

For the majority of this section, we will think of permutations in terms of the result of their application to the ordering \( \{1, 2, 3, \ldots, n\} \). Thus, \( \{2, 3, 1, 4, 5\} \) represents the permutation which has moved 2 to the first position, 3 to the second, 1 to the third, and left 4 and 5 fixed; that is, the permutation \((132)(4)(5)\) in cycle notation, or the inverse of the permutation \((12345)\), in two line notation.

We note that in order for \( vu^{-1} \) (or equivalently, \( v^{-1}u \)) to be a \( k \)-derangement, it is necessary and sufficient that no unordered \( k \)-tuple of elements be sent to the same unordered \( k \)-tuple of positions by both \( u \) and \( v \). For example, the permutation \( u = \{2, 3, 1, 4, 5\} \) and \( v = \{4, 1, 3, 5, 2\} \) both send the pair \( \{1,3\} \) to the second and third positions. Thus \((vu^{-1})(2)(\{2,3\}) = \{2,3\} \), and so \( vu^{-1} \) is not a 2-derangement and there is no edge between \( u \) and \( v \) in the 2-derangement graph. More formally, suppose \( u \) and \( v \) both send the \( k \)-tuple \( M = \{a_1, a_2, \ldots, a_k\} \) to positions \( M' = \{a'_1, a'_2, \ldots, a'_k\} \). Then, \((vu^{-1})(k)(M) = v(k)(M') = M \). Thus, \( vu^{-1} \) is not a \( k \)-derangement.

On the other hand, if \( u \) and \( v \) send no \( k \)-tuple to the same positions we claim \( vu^{-1} \) is a \( k \)-derangement. Consider an arbitrary \( k \)-tuple, \( M = \{a_1, a_2, \ldots, a_k\} \), and suppose \( u \) maps the \( k \)-tuple \( M' = \{a'_1, a'_2, \ldots, a'_k\} \) to the positions given in \( M \). Then \((vu^{-1})(k)(M) = v(k)(M') \neq M \) since \( v \) cannot send the \( k \)-tuple \( M' \) to the same positions as \( u \) does. Thus, \( vu^{-1} \) is a \( k \)-derangement.

In Theorem 6, we find the clique number of the 2-derangement graph, \( \omega(\Gamma_{2,n}) \), for \( n \) an odd prime power, by constructing a clique of maximal size. Before establishing this clique number, we note an upper bound on the clique number of a general \( k \)-derangement graph.

**Lemma 5.** For \( k < n \), \( \omega(\Gamma_{k,n}) \leq \binom{n}{k} \).
Proof. The clique number of the $k$-derangement graph, $\omega(\Gamma_{k,n})$ cannot be greater than $\binom{n}{k}$, since there are only $\binom{n}{k}$ subsets of size $k$ and hence at most $\binom{n}{k}$ different unordered $k$-tuples of positions for an arbitrary $k$-tuple of elements to be sent under a permutation. \hfill \Box

Theorem 6. If $n$ is an odd prime power, then $\omega(\Gamma_{2,n}) = \binom{n}{2}$.

Proof. We will explicitly construct a clique with $\binom{n}{2}$ elements. Let $n = p^k$ with $p$ a prime greater than 2, and let $\mathbb{F}_{p^k}$ denote the field with $p^k$ elements. Rather than letting $S_n$ act on $[n]$, we will let it act on $\mathbb{F}_{p^k}$ and construct $\Gamma_{2,n}$ accordingly. Let $v = (x_1, \ldots, x_n)$ be an ordered $n$-tuple whose entries are the elements of $\mathbb{F}_{p^k}$ in some order. Given any function $\phi : \mathbb{F}_{p^k} \rightarrow \mathbb{F}_{p^k}$, we define $\phi(v) = (\phi(x_1), \ldots, \phi(x_n))$. Partition the non-zero elements of $\mathbb{F}_{p^k}$ by pairing each element with its (additive) inverse, and let $T$ be a set obtained by choosing exactly one element from each pair, giving $|T| = (p^k - 1)/2$.

Define $f_{s,\alpha}(x) = sx + \alpha$, and consider the set $X = \{f_{s,\alpha}(v) | s \in T \text{ and } \alpha \in \mathbb{F}_{p^k}\}$. Since $s \neq 0$, $f_{s,\alpha}$ is a bijection and $f_{s,\alpha}(v)$ is a permutation of the elements of $\mathbb{F}_{p^k}$. We claim that $X$ is a clique in $\Gamma_{2,n}$. Suppose not; that is, suppose there are $s, t \in T$ and $\alpha, \beta \in \mathbb{F}_{p^k}$, $(s, \alpha) \neq (t, \beta)$, such that $f_{s,\alpha}(v)$ is not a 2-derangement of $f_{s,\beta}(v)$. In that case there exist $x, y \in \mathbb{F}_{p^k}$, $x \neq y$, such that either $f_{s,\alpha}(x) = f_{t,\beta}(x)$ and $f_{s,\alpha}(y) = f_{t,\beta}(y)$ or $f_{s,\alpha}(x) = f_{t,\beta}(y)$ and $f_{s,\alpha}(y) = f_{t,\beta}(x)$. In the first case, subtracting the two equations and rewriting yields $(s-t)(x-y) = 0$. If $s = t$, then $\alpha = \beta$ giving a contradiction. If $s \neq t$, then $x = y$ and again we have a contradiction. In the second case, subtracting and rewriting yields $(s+t)(x-y) = 0$ and since $s + t \neq 0$ for $s, t \in T$, $x = y$ and this also give a contradiction. Thus, $X$ is a clique of size $p^k((p^k - 1)/2) = \binom{n}{2}$.

\hfill \Box

Example 7. We build a clique of size $\binom{7}{2}$ in the derangement graph $\Gamma_{2,7}$ consisting of $\frac{7-1}{2}$ blocks, each of which contains 7 permutations. We let $v = (1, 2, 3, 4, 5, 6, 7)$ (writing 7 instead of 0) and take $T$ = \{1, 4, 5\}. Then $f_{1,0}(v) = (1, 2, 3, 4, 5, 6, 7)$, $f_{4,0}(v) = (4, 1, 5, 2, 6, 3, 7)$, and $f_{5,0}(v) = (5, 3, 1, 6, 4, 2, 7)$. Increasing $\alpha$ from 0 cyclically permutes the 7-tuples. Block 1 consists of the arrangements $\{f_{1,\alpha}(v) | \alpha \in \mathbb{F}_7\}$, that is the arrangement $(1, 2, 3, 4, 5, 6, 7)$ and the remaining 6 rotations of this arrangement (e.g., $(2, 3, 4, 5, 6, 7, 1)$, $(3, 4, 5, 6, 7, 1, 2)$, etc.). Block 2 consists of the arrangement $f_{4,0}(v)$ along with all of its rotations. Finally, Block 3 consists of $f_{5,0}(v)$ and its rotations. To see that these permutations form a clique, consider, for example, the pair $\{1, 2\}$. These elements are one position apart in block 1, two positions apart in block 2 and three positions apart in block 3 (counting the shortest distance between them either forwards or backwards). So the pair $\{1, 2\}$ cannot occupy the same positions in two permutations which appear in different blocks. Furthermore, within a block, the rotations insure that the pair never occupies the same positions.

Remark 8. The cliques which achieve the upper bound of Lemma 5 are known as sharply $k$-homogeneous sets of permutations. A corollary in [2] shows that for $2k \leq n$, the existence of such a $k$-homogeneous set implies $n+1 \equiv 0 \mod k$. Thus Theorem 6 cannot be extended to even $n$, and we have the following.
Corollary 9. For \( n \) even and \( n \geq 4 \), \( \omega(\Gamma_{2,n}) < \binom{n}{2} \).

A computer search confirms that \( \omega(\Gamma_{2,4}) = 5 < \binom{4}{2} \).

Next we turn to the independence number \( \alpha(\Gamma_{k,n}) \) and the chromatic number \( \chi(\Gamma_{2,n}) \) of the \( k \)-derangement graph. We will require the following lemma which has been adapted from Frankl and Deza’s lemma [1] and applied to \( k \)-tuples of elements.

Lemma 10. For \( k < n \), \( \alpha(\Gamma_{k,n}) \omega(\Gamma_{k,n}) \leq n! \).

Proof. Let \( \mathcal{P} \) be a set of permutations in \( S_n \), every pair of which has at least one unordered \( k \)-tuple of elements in the same unordered \( k \)-tuple of positions. That is, for any \( u, v \in \mathcal{P} \), there exists a set \( M = \{a_1, \ldots, a_k\} \subseteq [n] \) such that \( (v^{-1}u)_{(k)}(M) = M \). Note that \( \mathcal{P} \) is an independent set in the \( k \)-derangement graph. Let \( \mathcal{Q} \) be a set of permutations in \( S_n \) such that each pair of permutations has no \( k \)-tuple of elements in the same positions; that is, \( \mathcal{Q} \) is a clique in the \( k \)-derangement graph. We claim that products of the form \( PQ \) with \( P \in \mathcal{P} \) and \( Q \in \mathcal{Q} \) give distinct permutations of \( n \). Suppose, for the sake of contradiction, that \( P_1Q_1 = P_2Q_2 \) for \( P_1, P_2 \in \mathcal{P} \) and \( Q_1, Q_2 \in \mathcal{Q} \) with \( P_1 \neq P_2 \) and \( Q_1 \neq Q_2 \). This implies that \( P_1^{-1}P_2 = Q_1Q_2^{-1} \). Now, since \( P_1 \) and \( P_2 \) are in \( \mathcal{P} \), there is a \( k \)-tuple of elements \( M = \{a_1, \ldots, a_k\} \) such that \( (P_1^{-1}P_2)_{(k)}(M) = M \). However, this implies \( (Q_1Q_2^{-1})_{(k)}(M) = M \). But we know that the permutations in \( \mathcal{Q} \) agree on no \( k \)-tuples, and so we must have \( Q_1 = Q_2 \) and hence, \( P_1 = P_2 \). Finally, since each product gives a unique permutation of \( n \), there can be no more than \( n! \) such products. \( \square \)

Theorem 11. For \( k < n \), \( \alpha(\Gamma_{k,n}) \geq k!(n-k)! \) and \( \chi(\Gamma_{k,n}) \leq \binom{n}{k} \).

Proof. Consider \( H \), the set of all permutations in \( S_n \) that send \( \{1, 2, \ldots, k\} \) to itself (and hence \( \{k+1, \ldots, n\} \) to itself). It is clear that \( H \) is a subgroup of \( S_n \) isomorphic to \( S_k \times S_{n-k} \) and that \( |H| = k!(n-k)! \). Since the unordered \( k \)-tuple \( \{1, 2, \ldots, k\} \) is fixed, none of these are \( k \)-derangements of each other, so \( H \) is an independent set and \( \alpha(\Gamma_{k,n}) \geq k!(n-k)! \).

The cosets of \( H \) partition \( S_n \), and each forms an independent set, since \( \tau_1, \tau_2 \in \sigma H \) implies that \( \tau_1^{-1}\tau_2 \in H \) is not a \( k \)-derangement and hence the vertices associated to \( \tau_1 \) and \( \tau_2 \) are not connected by an edge. Giving each of the \( \frac{n!}{k!(n-k)!} \) cosets a different color results in a valid coloring of \( \Gamma_{k,n} \), so \( \chi(\Gamma_{k,n}) \leq \binom{n}{k} \). \( \square \)

Corollary 12. For \( n \) an odd prime power, \( \alpha(\Gamma_{2,n}) = 2(n-k)! \) and \( \chi(\Gamma_{2,n}) = \binom{n}{2} \).

Proof. By Lemma 10 and Theorem 6, we have \( \binom{n}{2} \cdot \alpha(\Gamma_{2,n}) \leq n! \). Thus \( \alpha(\Gamma_{2,n}) \leq n! \cdot \frac{2(n-2)!}{n!} = 2(n-2)! \) and Theorem 11 gives the reverse inequality. For any graph \( G \), \( \chi(G) \geq \omega(G) \), so by Theorem 6, \( \chi(\Gamma_{2,n}) \geq \binom{n}{2} \) and again Theorem 11 gives the reverse inequality. \( \square \)

5. Further Questions

In the last section, we showed that the clique number of the 2-derangement graph is equal to \( \binom{n}{2} \) when \( n \) is an odd prime power and strictly less than that if \( n \) is even (and at least 4). The clique construction of Theorem 6 fails to work when \( n \) is odd and not a prime power since there is no field of that cardinality. We believe that in this case the
clique number is strictly smaller than \( \binom{n}{2} \). For arbitrary \( k \), we have some faint hope that the bounds given in Theorem 11 for \( \alpha(\Gamma_{k,n}) \) and \( \chi(\Gamma_{k,n}) \) are actually equalities, but the situation for \( \omega(\Gamma_{k,n}) \) remains unclear.

In another direction, the numerical evidence is overwhelming that the derangement graphs are Hamiltonian. We hope to explore these and other questions in future work.

**Acknowledgements**

The first author worked on this topic with the third author at an REU at Missouri State University in the summer of 2009 (award #0552573). The authors would like to acknowledge Sam Tencer’s contribution to this investigation. The authors are also grateful to an anonymous referee for spotting the connection between cliques attaining the bound of Lemma 5 and sharply \( k \)-homogeneous sets of permutations.

**References**


Mathematics Department, Syracuse University, 215 Carnegie, Syracuse, New York 13244 U.S.A.

*E-mail address:* hljacksc@syr.edu

Mathematics Department, Willamette University, 900 State Street, Salem, Oregon 97301 U.S.A.

*E-mail address:* knyman@willamette.edu

Mathematics Department, Missouri State University, 901 South National Avenue, Springfield, Missouri 65897 U.S.A.

*E-mail address: LesReid@MissouriState.edu*