The Cauchy-Schwartz inequality for integrals (group of three).

(See chapter 6.3 #16) Let \( f, g \in \mathbb{R}[a,b] \). Prove that

\[
\left| \int_a^b f \cdot g \right|^2 \leq \left( \int_a^b f^2 \right) \left( \int_a^b g^2 \right)
\]

(Hint: For \( \alpha, \beta \in \mathbb{R} \) consider \( \int_a^b (\alpha f - \beta g)^2 \))

Well, that’s all the hint the text gives. It’s suggesting a method you might have seen in your linear algebra course to prove “the Cauchy-Schwartz inequality” (for vectors). For perspective, in Anton & Rorres’ linear algebra text, before using this method they say “We warn the reader in advance that the proof presented here depends on a clever trick that is not easy to motivate.” So I’ll give a few more hints in the following outline:

1. Let \( f, g \in \mathbb{R}[a,b] \). It is important that \( f \) and \( g \) will not change over the course of this proof. Explain why that implies that \( \int_a^b f^2, \int_a^b f \cdot g, \text{ and } \int_a^b g^2 \) are all constants. Call them \( A, B, \) and \( C \) respectively. PS - why are each of these integrals defined?

2. Why is \( \int_a^b (\alpha f - \beta g)^2 \geq 0 \) for all \( \alpha, \beta \in \mathbb{R} \)?

3. Expand \( \int_a^b (\alpha f - \beta g)^2 \) until you can replace \( \int_a^b f^2, \int_a^b f \cdot g, \text{ and } \int_a^b g^2 \) by \( A, B, \text{ and } C \) respectively.

4. Combine (2) and (3) to write an inequality that involves only \( \alpha, \beta, A, B, \text{ and } C \) (well, and “2”).

5. Thinking of \( \alpha \) as a variable, and all the remaining symbols as constants, the graph of the (nonzero side of the) inequality has what shape? Geometrically, what does the inequality tell us about the graph? How many roots does it have?

6. So the discriminant of the (nonzero side of the) inequality has what sign? Use this to get the desired inequality.

7. Bonus: Up to now, we could have used 1 in the place of \( \beta \), but our proof above would break down in one case which the \( \beta \) will patch up. Can you find it?

A triangle inequality for integrals (group of two).

Define the norm of \( f \), denoted \( \| f \| \), for all functions \( f \in \mathbb{R}[a,b] \) by

\[
\| f \| = \sqrt{\int_a^b f^2},
\]

Then prove that if \( f, g, h \in \mathbb{R}[a,b] \), we have

\[
\| f - h \| \leq \| f - g \| + \| g - h \|,
\]

that is,

\[
\sqrt{\int_a^b (f - h)^2} \leq \sqrt{\int_a^b (f - g)^2} + \sqrt{\int_a^b (g - h)^2}.
\]

Note: Do explain why \( \| f \|, \| f - h \|, \| f - g \|, \text{ and } \| g - h \| \) are even defined.

Hint: Work with \( \int_a^b (f - h)^2 \) and (after a fair bit of other work,) use the Cauchy-Schwartz inequality. At some point, things may get clearer if you use norm notation; recall \( \int_a^b (f - h)^2 \) is \( \| f - h \|^2 \) (don’t forget the second square).
An equivalence relation (group of three).

Define the relation ∼ on \( \mathcal{R} [a, b] \), where ∼ is defined by \( f \sim g \) iff \( \int_a^b (f - g)^2 = 0 \).

1. Tell why \( \int_a^b (f - g)^2 \) is always defined, so that we can determine whether or not it is zero for each \( f, g \in \mathcal{R} [a, b] \).

2. Prove that ∼ is indeed an equivalence relation on \( \mathcal{R} [a, b] \). Transitivity is tricky. One of the earlier results on this handout will be useful.

3. Review for us what an equivalence class for this relation would be (this is Foundations stuff). In particular, remembering that the equivalence class of \( f \) is denoted \( [f] \), then give us several members of \( [0] \), where \( 0 \) is the zero function (constantly equal to zero). You may assume \( a \) and \( b \) are any specific numbers you like for the purposes of this example. Hint: Last take-home: (somethings) “don’t matter.” Try to go beyond the obvious examples. You may need (4) below to finish (3), depending on the order in which you do things.

4. Remind us why \( [f] = [g] \) if and only if \( f \sim g \). (Give a proof, straight out of Foundations.)

5. If it helps, the point of this exercise (beyond setting up the next problem) is to generate a new kind of “equality” between functions. This will enable us to deal usefully with functions which perhaps “are not actually fully equal, but are equal enough that integration can’t tell the difference between them.”

An integration metric (group of two)

Show that the function given by \( d ([f], [g]) = \| f - g \| \) is a metric on the equivalence classes of ∼ for \( \mathcal{R} [a, b] \), where ∼ is defined as in the problem above. Pay attention to the fussy bits here. In particular,

1. Show that if \( f, \hat{f} \in [f] \) and \( g, \hat{g} \in [g] \), then \( \| f - g \| = \| \hat{f} - \hat{g} \| \), so that which elements of \( [f] \) and \( [g] \) we use in “\( \| f - g \| \)” don’t matter. (This shows that \( d ([f], [g]) \) is well-defined).

2. Use earlier results from this handout whenever possible, especially the prior problem.