1. Let \( f(x) = x^3 - 3x + 1 \). Since \( f \) is a polynomial, it is differentiable and continuous everywhere, including \([1, 3] \). Therefore, Rolle’s theorem applies and \( f'(c) = 0 \) for some \( c \in (1, 3) \). Specifically, \( f'(x) = 3x^2 - 3 \), and this is zero for \( x = 1 \). Thus, \( c = 2 \).

5. Let \( f(1) = 0 = f(1) \), but \( f'(x) = -\frac{2}{3x^{1/3}} \) is never zero on \((-1, 1)\). Although \( f \) is continuous on \([-1, 1] \), \( f \) is not differentiable on \((-1, 1)\) since \( f'(x) \) is undefined at \( x = 0 \). Thus, there is no contradiction to Rolle’s Theorem.

9. See the back of the book for the graphs. For part (c), \( f'(x) = 1 - \frac{4}{x^2} \). We need to know when this is equal to \( \frac{f(8) - f(1)}{8 - 1} = 0.5 \). We solve \( 1 - \frac{4}{x^2} = \frac{1}{2} \). \( 2x^2 = 8 \), so \( x^2 = 8 \). Thus \( x = \sqrt{8} = 2\sqrt{2}. \) (We ignore the negative root since it does not lie in \([1, 8]\).)

14. Since \( f \) is a rational function, it is differentiable and continuous on its domain, which includes \([1, 4]\).
\[
f'(x) = \frac{x + 2 - x}{(2 + x)^2} = \frac{2}{(x + 2)^2}.
\]
The slope of the secant line is \( \frac{f(4) - f(1)}{4 - 1} = \frac{2/3 - 1/3}{3} = \frac{1}{9} \). We solve \( \frac{2}{(x + 2)^2} = \frac{1}{9} \) for \( x \): \( 18 = (x + 2)^2 \), so \( x + 2 = \pm 3\sqrt{2} \) and \( x = \pm 3\sqrt{2} - 2 \). Only the positive root is in the interval, so \( c = 3\sqrt{2} - 2 \).

17. \( f(-1) = -6 \) and \( f(0) = 1 \), so \( f \) has a root \( x_1 \) in \((-1, 0)\) by the Intermediate Value Theorem. \( f'(x) = 20x^4 + 3x^2 + 2 \geq 2 \), so \( f'(x) \) is never zero. If \( f \) had another root \( x_2 \), then Rolle’s Theorem would imply that \( f' \) had a zero in \((x_1, x_2)\), which is not the case. Therefore, \( f \) cannot have a second root, so it has exactly one.

20. If \( f(x) = x^4 + 4x + c \), then \( f'(x) = 4x^3 + 4 = 4(x + 1)(x^2 - x + 1) \). If \( f \) had three distinct zeros, then \( f' \) would have to have at least 2 zeros by Rolle’s Theorem. However, the only zero of \( f' \) is \( x = -1 \). Therefore, \( f \) has at most two distinct zeros.

21. (a) Let \( P \) be a polynomial of degree 3. If \( P \) has four roots, then between each consecutive pair will lie a zero of the derivative by Rolle’s Theorem. But the derivative \( P' \) is a polynomial of degree 2; if it has three zeros, then between each consecutive pair will lie a zero of \( P'' \), again by Rolle’s Theorem applied to \( P' \). This means that \( P'' \) will have 2 zeros. Now \( P'' \) is a polynomial of degree 1, which we know has at most one zero, so this is not possible. Therefore, \( P \) cannot have 4 distinct zeros.

(b) If our polynomial of degree \( n \) is \( P \), then for \( P \) to have \( n + 1 \) zeros would require that \( P' \) have at least \( n \) zeros, as above. But \( P' \) is a polynomial of degree \( n - 1 \), and we may continue backward as we did in (a) to find that a polynomial of degree 1 would have two zeros.

23. Since \( f \) is differentiable on \([1, 4]\), it is automatically continuous on \([1, 4]\), so the Mean Value Theorem applies. Whatever \( f(4) \) is, there will be a \( c \in (1, 4) \) such that \( f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{f(4) - 10}{3} \), so \( f(4) = 3f'(c) + 10 \geq 3(2) + 16 \). Thus, \( f(4) \geq 16 \).

26. Let \( h(x) = f(x) - g(x) \). Then \( h \) is continuous on \([a, b]\) and differentiable on \((a, b)\) since \( f \) and \( g \) are. Note that \( h(a) = 0 \). By the Mean Value Theorem, there is a number \( c \in (a, b) \) such that \( h'(c) = \frac{h(b) - h(a)}{b - a} = \frac{h(b)}{b - a} = \frac{f'(c) - g'(c)}{b - a} \). Since \( f'(c) < g'(c) \), the numerator is negative. Therefore, \( h(b) = f'(c) - g'(c) < 0 \), so \( f(b) - g(b) < 0 \). This is what we were hoping for: \( f(b) < g(b) \).

34. Let \( h(x) = f(x) - x \). If \( f \) has two fixed points \( a \) and \( b \), then \( h(a) = 0 = h(b) \), so \( h \) has a derivative of zero somewhere between \( a \) and \( b \). But \( h'(x) = f'(x) - 1 \), which is never zero! Therefore \( f \) cannot have two fixed points.