Math 249 Exam III

Tuesday, November 16, 2004

Remember to show all work. Unsupported solutions will receive no credit.

1. (20 points) Consider \( f(x, y) = e^{xy} \).
   (a) Find \( \nabla f(x, y) \).
   (b) Sketch the level curves for \( z = e^{-1}, 1, e \). Sketch as accurately as possible and label each.
   (c) Draw the gradient at \((1, 1)\) on your level curves.
   (d) What does \( \nabla f(1, 1) \) mean? (I’m not looking for a name; I’m looking for an interpretation!)
   (e) Find the directional derivative of \( f \) in the direction of \( \langle -3, 4 \rangle \) at \((1, 1)\). How does this number compare with the length of \( \nabla f(1, 1) \)? Why?

2. (10 points) As a car climbs a mountain, a scientist is measuring atmospheric pressure. She determines that the pressure at \((x, y)\) (coordinates on a map) is given by \( P(x, y) = 70 + \frac{3}{x^7y^2} \) kPa. If \( x(t) = (10 - t) \cos(2\pi t) \) and \( y(t) = (10 - t) \sin(2\pi t) \), where \( t \) is measured in hours, find \( \frac{dP}{dt} \) at \( t = 2 \) hours.

3. (10 points) A toy manufacturer has a mold to create a plastic “Rapunzel’s Tower,” which is a cone placed on top of a cylinder. The inner radius of each is 5 cm, the inside heights of the cylinder and cone are each 15 cm. The cylinder has a bottom. If the plastic is 0.15 cm thick, use differentials to estimate the amount of plastic in each Rapunzel’s Tower.

4. (20 points) Find the critical points of \( f(x, y) = 2x^2 - 4xy + y^3 + 2 \) and classify each as a local maximum, local minimum, or neither.

5. (10 points) Compute \( \iint_D xe^{xy} \, dA \) if \( D = [0, 1] \times [-1, 1] \).

6. (10 points) Compute \( \iint_D \frac{1}{\sqrt{4 - x^2 - y^2}} \, dA \), where \( D = \{(x, y)|x \geq 0, y \geq 0, \text{ and } 1 \leq x^2 + y^2 \leq 4\} \).

7. (10 points) Compute \( \int_0^1 \int_{2x}^1 e^{-y^2} \, dy \, dx \) by reversing the order of integration. Sketch the region of integration.

8. (10 points) True or False.
   (a) If \( L \) is the linearization of \( f \) at \((x_0, y_0)\), then the graph of \( L \) is the tangent plane to the graph of \( f \) at \((x_0, y_0)\).
   (b) For any function \( f \) of two variables, \( f_{xy} = f_{yx} \).
   (c) If \( f \) is continuous on the bounded set \( D \), then \( f \) has an absolute maximum and minimum on \( D \).
   (d) If \( f \) is continuous on \([a, b] \times [c, d]\), then \( \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy \).
   (e) If \( f \) is a function of two variables, then \( \nabla f(x_0, y_0) \) is perpendicular the tangent plane to the graph of \( f \) at \((x_0, y_0)\).