Math 249 Exam III

Tuesday, November 16, 2004

Remember to **show all work**. Unsupported solutions will receive **no credit**.

1. (20 points) Consider \( f(x, y) = e^{xy} \).
   (a) Find \( \nabla f(x, y) \).
   **Solution:** \( \nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \langle ye^{xy}, xe^{xy} \rangle \).
   (b) Sketch the level curves for \( z = e^{-1}, 1, e \). Sketch as accurately as possible and label each.
   **Solution:** (for both (b) and (c))
   (c) Draw the gradient at \((1, 1)\) on your level curves.
   (d) What does \( \nabla f(1, 1) \) mean? (I’m not looking for a name; I’m looking for an interpretation!)
   **Solution:** This is the direction of greatest increase on the graph of \( f \) at the point \((1, 1)\).
   (e) Find the directional derivative of \( f \) in the direction of \( \langle -3, 4 \rangle \) at \((1, 1)\). How does this number compare with the length of \( \nabla f(1, 1) \)? Why?
   **Solution:** \( D_u f(1, 1) = \langle e, e \rangle \cdot \langle -3, 4 \rangle = e \frac{5}{3} \). It is smaller than the length of \( \nabla f(1, 1) \) since the latter is the maximum rate of increase (for any direction) at \((1, 1)\).

2. (10 points) As a car climbs a mountain, a scientist is measuring atmospheric pressure. She determines that the pressure at \((x, y)\) (coordinates on a map) is given by \( P(x, y) = 70 + 3x^2 + y^2 \) kPa. If \( x(t) = (10-t)\cos(2\pi t) \) and \( y(t) = (10-t)\sin(2\pi t) \), where \( t \) is measured in hours, find \( \frac{dP}{dt} \) at \( t = 2 \) hours.
   **Solution:**
   \[
   \frac{dP}{dt} = \frac{dP}{dx} \frac{dx}{dt} + \frac{dP}{dy} \frac{dy}{dt} = \frac{3x}{\sqrt{x^2 + y^2}}(\cos(2\pi t) - 2\pi(10-t)\sin(2\pi t)) + \frac{3y}{\sqrt{x^2 + y^2}}(-\sin(2\pi t) - 2\pi(10-t)\cos(2\pi t)).
   \]
   At \( t = 2 \), we have \( x = 8 \) and \( y = 0 \), so \( \frac{dP}{dt} = -3 \).

3. (10 points) A toy manufacturer has a mold to create a plastic “Rapunzel’s Tower,” which is a cone placed on top of a cylinder. The inner radius of each is 5cm, the inside heights of the cylinder and cone are each 15 cm. The cylinder has a bottom. If the plastic is 0.15 cm thick, use differentials to estimate the amount of plastic in each Rapunzel’s Tower.
   **Solution:** \( V = \frac{1}{3} \pi r^2 h + \pi r^2 h \), so \( dV = \frac{4}{3}(2\pi rhdr + \pi r^2 dh) = \frac{4}{3}(2\pi(5)(15)(0.15) + \pi(5^2)(0.15) = 35\pi \approx 110 \text{ cm}^3 \).
4. (20 points) Find the critical points of \( f(x, y) = 2x^2 - 4xy + y^3 + 2 \) and classify each as a local maximum, local minimum, or neither.

**Solution:** \( f_x = 4x - 4y, f_y = -4x + 3y^2 \). From \( f_x = 0 \), we get \( y = x \), so \( f_y = 0 \) implies \( 3y^2 = 4y \). Thus \( y = 0 \) or \( y = 4/3 \). The critical points are \((0, 0)\) and \((4/3, 4/3)\). (Recall we have \( y = x \).)

Now \( f_xx = 4, f_xy = -4 \), and \( f_yy = 6y \), so \( D(x, y) = 24y - 16 \). \( D(0, 0) = -16 \), so \((0, 0)\) is a saddle point. \( D(4/3, 4/3) = 16 > 0 \). \( f_xx(4/3, 4/3) = 4 > 0 \), so we have a local minimum at \((4/3, 4/3)\).

5. (10 points) Compute \( \iint_D xe^{xy} dA \) if \( D = [0, 1] \times [-1, 1] \).

**Solution:**

\[
\iint_D xe^{xy} dA = \int_0^1 \int_{-1}^1 xe^{xy} dy dx = \int_0^1 (e^x - e^{-x}) dx = e^x + e^{-x} \Big|_0^1 = e + e^{-1} - 2.
\]

6. (10 points) Compute \( \iint_D \frac{1}{\sqrt{4 - x^2 - y^2}} dA \), where \( D = \{(x, y)|x \geq 0, y \geq 0, \text{ and } 1 \leq x^2 + y^2 \leq 4\} \).

**Solution:**

\[
\iint_D \frac{1}{\sqrt{4 - x^2 - y^2}} dA = \int_0^{\pi/2} \int_1^2 \frac{r}{\sqrt{4 - r^2}} dr d\theta = \frac{\pi}{2} \left(-\sqrt{4 - r^2}\right) \Big|_1^2 = \frac{\pi \sqrt{3}}{2}.
\]

7. (10 points) Compute \( \int_0^1 \int_{2x}^2 e^{-y^2} dy dx \) by reversing the order of integration. Sketch the region of integration.

**Solution:** The region of integration is the triangle shown below.

We get

\[
\int_0^1 \int_{2x}^2 e^{-y^2} dy dx = \int_0^2 \int_0^{y/2} e^{-y^2} dx dy = \int_0^y \frac{y}{2} e^{-y^2} dy = \left[-\frac{1}{4} e^{-y^2}\right]_0^2 = 1 - e^{-4}.
\]
8. (10 points) True or False.

(a) **TRUE** If $L$ is the linearization of $f$ at $(x_0, y_0)$, then the graph of $L$ is the tangent plane to the graph of $f$ at $(x_0, y_0)$.

(b) **FALSE** For any function $f$ of two variables, $f_{xy} = f_{yx}$.

(c) **FALSE** If $f$ is continuous on the bounded set $D$, then $f$ has an absolute maximum and minimum on $D$.

(d) **TRUE** If $f$ is continuous on $[a, b] \times [c, d]$, then $\int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy$.

(e) **FALSE** If $f$ is a function of two variables, then $\nabla f(x_0, y_0)$ is perpendicular the tangent plane to the graph of $f$ at $(x_0, y_0)$. 