Math 251-02/03 Homework 10

April 15, 2012

Rough drafts due 4/16/12, edits due 4/18/12, final drafts due 4/20/12.

Problems to Keep:

1. 6.3.1 (2,3) Prove the formulas hold for all $n \in \mathbb{N}$.

   (2) $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

   **Base Case:** If $n = 1$, the left-hand side is 1 and the right-hand side is also 1, so the Base Case holds.

   **Induction Step:** Now assume that for some $n \in \mathbb{N}$, $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$. We want to show that $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$.

   We calculate:

   \[
   1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2
   \]

   \[
   1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (n+1) \left( \frac{n(2n+1)}{6} + (n+1) \right)
   \]

   \[
   1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (n+1) \left( \frac{2n^2 + n + 6n + 6}{6} \right)
   \]

   \[
   1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (n+1) \left( \frac{2n^2 + 7n + 6}{6} \right)
   \]

   \[
   1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (n+1) \left( \frac{(n+2)(2n+3)}{6} \right)
   \]

   \[
   1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6},
   \]

   as desired. Therefore, by the Principle of Mathematical Induction, $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

   (3) $1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$.

   **Base Case:** If $n = 1$, the left-hand side is 1 and the right-hand side is also 1, so the Base Case holds.
**Induction Step:** Now assume that for some \( n \in \mathbb{N}, \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}. \)

We want to show that \( \sum_{k=1}^{n} k^3 + (n+1)^3 = \frac{(n+1)^2(n+2)^2}{4}. \) We calculate:

\[
\begin{align*}
1^3 + 2^3 + \ldots + n^3 &= \frac{n^2(n+1)^2}{4} \\
1^3 + 2^3 + \ldots + n^3 + (n+1)^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
1^3 + 2^3 + \ldots + n^3 + (n+1)^3 &= (n+1)^2 \left( \frac{n^2}{4} + n + 1 \right) \\
1^3 + 2^3 + \ldots + n^3 + (n+1)^3 &= (n+1)^2 \left( \frac{n^2 + 4n + 4}{4} \right) \\
1^3 + 2^3 + \ldots + n^3 + (n+1)^3 &= (n+1)^2 \left( \frac{(n+2)^2}{4} \right) \\
1^3 + 2^3 + \ldots + n^3 + (n+1)^3 &= \frac{(n+1)^2(n+2)^2}{4},
\end{align*}
\]

as desired. Therefore, by the Principle of Mathematical Induction, \( \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \) for all \( n \in \mathbb{N}. \)

2. **6.3.2** Prove that \( 1 + 2n \leq 3^n \) for all \( n \in \mathbb{N}. \)

**Base Case:** If \( n = 1, \) then \( 1 + 2n = 1 + 2(1) = 3 \leq 3^1 = 3^n, \) so the base case holds.

**Induction Step:** Assume that for some \( n \geq 1, \) \( 1 + 2n \leq 3^n. \) We want to show that \( 1 + 2(n+1) \leq 3^{n+1}, \) which is to say that \( 3 + 2n \leq 3^{n+1}. \)

Since \( 1 + 2n \leq 3^n, \) we know that \( 3 + 2n \leq 3^n + 2. \) But \( 2 \leq 2 \cdot 3^n \) since \( n \geq 1, \) so \( 3 + 2n \leq 3^n + 2 \cdot 3^n = 3^{n+1}, \) as desired.

Therefore, by the Principle of Mathematical Induction, \( 1 + 2n \leq 3^n \) for all \( n \in \mathbb{N}. \)

3. **6.3.6** For which values of \( n \) does \( n^2 - 9n + 19 > 0 \) hold?

Completing the square gives \( n^2 - 9n + 19 = (n - 4.5)^2 - 12.25 = (n - 4.5)^2 - 1.25. \) For this to be greater than 0, we need \( n \geq 6 \) or \( n \leq 3. \) It is easy to check that the inequality holds for \( n = 1, 2, 3 \) and fails for \( n = 4, 5. \)

**Base Case:** \( n = 6: \) The inequality is easily seen to hold, so the base case is established.

**Induction Step:** Assume that for some \( n \in N, \) \( n^2 - 9n + 19 > 0. \) We want to show that \( (n+1)^2 - 9(n+1) + 19 > 0. \) Compute:

\[
\begin{align*}
(n+1)^2 - 9(n+1) + 19 &= (n^2 + 2n + 1) + (-9n - 9) + 19 \\
&= (n^2 - 9n + 19) + (2n - 8) \\
&> 2n - 8 \\
&\geq 0
\end{align*}
\]

since \( n \geq 6 \) implies \( 2n - 8 \geq 0. \) Note the use of the Induction Hypothesis in going from line 2 to line 3.
Therefore \((n + 1)^2 - 9(n + 1) + 19 > 0\), so by the Principle of Mathematical Induction, \(n^2 - 9n + 19 > 0\) for all \(n \geq 6\).

4. **6.3.12** Prove that \(\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}\) for all \(n \geq 2\).

**Base Case:** If \(n = 2\), the left-hand side is \(1 + \frac{1}{2} = 3/2\), while the right-hand side is \(\sqrt{2} \approx 1.414\), so the inequality holds. This establishes the base case.

**Induction Step:** Now assume that for some \(n \in \mathbb{N}\), \(\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}\). We want to show that \(\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} > \sqrt{n+1}\).

We calculate:

\[
\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \left(\sum_{i=1}^{n} \frac{1}{\sqrt{i}}\right) + \frac{1}{\sqrt{n+1}} > \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n}\sqrt{n+1} + 1}{\sqrt{n+1}} > \frac{n + 1}{\sqrt{n+1}} = \sqrt{n+1}.
\]

Note that \(\sqrt{n}\sqrt{n+1} > n\) since \(\sqrt{n+1} > \sqrt{n}\).

Therefore \(\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} > \sqrt{n+1}\), so by the Principle of Mathematical Induction, \(\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}\) for all \(n \in \mathbb{N}\).