Solutions to Homework Assignment 13

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1. (a) \( u = 2x - 2xy \), so \( u_x = 2 - 2y \) and \( u_{xx} = 0 \). Also, \( u_y = -2x \) and \( u_{yy} = 0 \), so \( u \) is harmonic. If \( v \) is a harmonic conjugate of \( u \), then \( v_y = u_x = 2 - 2y \), so \( v = 2y - y^2 + g(x) \) for some function \( g \). Then \( v_x = g'(x) = -u_y = 2x \), so \( g(x) = x^2 + C \). Thus \( v(x, y) = 2y - y^2 + x^2 \) is a harmonic conjugate of \( u \).

(b) \( u_{xx} = -6x \) and \( u_{yy} = 6x \), so \( u \) is harmonic. If \( v \) is a harmonic conjugate of \( u \), then \( v_y = u_x = 2 - 3x^2 + 3y^2 \), so \( v = 2y - 3x^2y + y^3 + g(x) \) for some function \( g \). Thus \( v_x = -u_y = -6xy \), so \( g'(x) = 0 \). Thus \( v = 2y - 3x^2y + y^3 \) is a harmonic conjugate of \( u \).

(c) (c) and (d) are similar.

3. Since \( v \) is a harmonic conjugate of \( u \) and vice-versa, we have \( u_x = v_y \) and \( u_y = -v_x \) as well as \( v_x = u_y \) and \( v_y = -u_x \). Thus \( u_x = -u_x \), so \( u_x = 0 \). This also gives us \( v_y = 0 \), and we can likewise find \( u_y = v_x = 0 \). Since all partials are zero, \( u \) and \( v \) must both be constant.

9. We have \( f(z) = \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} \). Thus the level curves to plot are \( \frac{x}{x^2 + y^2} = C_1 \) and \( \frac{-y}{x^2 + y^2} = C_2 \) for values of \( C_1 \) and \( C_2 \). The graphs are below.