1. (a) We know \( b = ax \) for some integer \( x \in \mathbb{Z} \). Thus \( bc = (ax)c = a(xc) \), so \( a|bc \).

(b) If \( a|b \), then \( b = ax \) for some \( x \). Thus \( bc = axc = (ac)x \), so \( ac|bc \). Conversely, if \( ac|bc \), then we have \( bc = (ac)x \) for some \( x \in \mathbb{Z} \). Since \( c \neq 0 \), \( b = ax \), and thus \( a|b \).

(c) With \( b = ax \) and \( d = cy \), we have \( bd = (ac)(xy) \), so \( ac|bd \).

3. (a) If \( 3|a \), we are done. If not, then \( a = 3q + 1 \) or \( a = 3q + 2 \). Thus \( 2a^2 + 7 = 2(9q^2 + 9q + 1) + 7 = 9(2q^2 + 2q + 1) \), or \( 2a^2 + 7 = 2(9q^2 + 12q + 4) + 7 = 3(2q^2 + 8q + 5) \). In either case, \( 3|2a^2 + 7 \) and hence \( 3|a(2a^2 + 7) \).

(b) If \( a \) is odd, then \( a \) has the form \( 2a + 1 \). Thus \( (a^2 + 3)(a^2 + 7) = (4a^2 + 4a + 4)(4a^2 + 4a + 8) = 16(a^2 + a + 1)(a^2 + a + 2) \). Since \( a^2 + a + 1 \) and \( a^2 + a + 2 \) are consecutive integers, one of them is even. Thus \( (a^2 + a + 1)(a^2 + a + 2) = 2k \) for some integer \( k \), so \( (a^2 + 3)(a^2 + 7) = 32k \), and \( 32|(a^2 + 3)(a^2 + 7) \), as desired.

4. If \( a > 0 \) and \( a|1 \), then \( 1 = ax \) for some integer \( x \), which must positive since \( ax \) is positive. But \( a > 1, x \geq 1 \implies 1 = ax > 1 \), a contradiction, so \( a = 1 \). If \( a < 0 \), then \( -a > 0 \) and we find that \( -a = 1 \), so \( a = -1 \).

5. If \( a|b \) and \( b|a \), then we have \( b = ax = (by)x \) for some integers \( x, y \). Since \( b|0, b \neq 0 \), so we find \( xy = 1 \). By part (a), \( x = \pm 1 \), so \( b = \pm a \).

7. \( 1364 = 80 \cdot 17 + 4 \).

8. (a) Let \( n \in \mathbb{Z} \). Then by the division algorithm, \( n = 3q, 3q + 1, \) or \( 3q + 2 \) for some \( q \). Thus \( n^2 = 9q^2 = 3(3q^2), n^2 = (3q + 1)^2 = 9q^2 + 9q + 1 = 3(3q^2 + 3q) + 1, \) or \( n^2 = (3q + 2)^2 = 9q^2 + 12q + 4 = 3(3q^2 + 4q + 1) + 1 \). Thus, \( n^2 \) is of the form \( 3k \) or \( 3k + 1 \).

(b) If \( n \in \mathbb{Z} \), then \( n \) is of the form \( 9q, 9q + 1, \ldots, 9q + 8 \). \( 9q, 9q + 3, \) and \( 9q + 6 \) all have cubes that are a multiple of \( 3 \), so in these cases, \( n^3 \) is of the form \( 9k \). The cubes of \( 9q + 1, 9q + 4, \) and \( 9q + 7 \) all have the form \( 9q + 1 \), and the cubes of \( 9q + 2, 9q + 5, \) and \( 9q + 8 \) all have the form \( 9q + 8 \).