Prop.: For all real numbers $a$, one and only one of the following holds:

- $a$ is a positive number
- $a$ is zero
- $a$ is a negative number

**Proof**

Let $a$ be a real number. We will prove this is four cases.

**Case 1:** Suppose $a$ is positive. Thus, $a$ cannot be zero, by part 2 of the order axiom. Now suppose $a$ is positive and negative. Thus, $a \in \mathbb{R}^+$ and $a \in \mathbb{R}^-$. By definition of the negative numbers, this means that $a \notin \mathbb{R}^+$. Thus, we have a contradiction. So $a$ cannot be both positive and negative. Thus, if $a$ is positive, it is neither zero nor negative.

**Case 2:** Suppose $a$ is zero. By part 2 of the order axiom, $a$ is not positive. Now suppose $a = 0$ and $a$ is negative. Thus, $a \in \{0\}$ and $a \in \mathbb{R}^-$. By definition of the negative numbers, $a \notin \mathbb{R}^+ \cup \{0\}$. Thus, we have a contradiction. So if $a = 0$, $a$ is neither negative nor positive.

**Case 3:** Suppose $a$ is a negative number. By definition of negative numbers, $a \notin \mathbb{R}^+ \cup \{0\}$. Thus, $a$ is neither positive nor zero.

**Case 4:** Suppose $a$ is not positive, not zero, and not negative. Since $a$ is not negative, by definition of a negative number, $a \in \mathbb{R}^+ \cup \{0\}$. Thus, we have a contradiction, since $a$ is not positive or zero. So $a$ must be either positive, negative, or zero. ■