Let \((s_i)\) be a sequence in \(A\). Prove that \((s_i)\) has a constant subsequence if and only if there exists \(a \in A\) such that for all \(n \in \mathbb{N}\) there exists \(j \geq n\) such that \(s_j = a\).

**Proof**

(\(\Rightarrow\)) Suppose \((s_i)\) has a constant subsequence. By definition 0.4.3(2), there exists an \(a \in A\) such that \(s_{n_i} = a\) for all \(n_i \in \mathbb{N}\). Let \(s_k\) be an arbitrary element of \((s_i)\). Since by definition 0.4.9, \((n_i)\) is a strictly increasing sequence, there exists an \(n_l \geq k\) such that \(s_{n_l} = a\). Thus, for all \(n \in \mathbb{N}\), there exists a \(j \geq n\) such that \(s_j = a\), since \(k\) was arbitrary.

(\(\Leftarrow\)) Suppose there exists an \(a \in A\) such that for all \(n \in \mathbb{N}\) there exists a \(j \geq n\) such that \(s_j = a\). We will construct a constant subsequence of \((s_i)\) by induction.

**Base case:** Let \(s_{n_1}\) be the first term such that \(s_i = a\) in \((s_i)\). Thus, \(s_{n_1} = a\).

**Induction hypothesis:** Assume the first \(k\) terms have been chosen for the subsequence \((s_{n_i})\) such that \(s_{n_i} = a\) and \(n_{i+1} > n_i\) for all \(i \in \mathbb{N}\).

**Induction step (part I):** Consider \(s_{n_{k+1}}\). Since \(s_{n_{k+1}} \in (s_i)\), there exists an \(n_{k+1} \geq n_k + 1\) such that \(s_{n_{k+1}} = a\). So there exists an \(n_{k+1} > n_k\) such that \(s_{n_{k+1}} = a\). So \(s_{n_{k+1}}\) is the next term in the subsequence.

**Induction step (part II):** Let \((s_{n_i}) = \{s_{n_1}, s_{n_2}, \ldots, s_{n_k}, s_{n_k+1}, \ldots\}\). Thus, for all \(n_i \in \mathbb{N}, n_{i+1} > n_i\) and \(s_{n_i} = a\). This satisfies the conditions in the induction hypothesis, so \((s_{n_i})\) is a constant subsequence of \((s_i)\). ■