Theorem 1.3.8 Part 3

Proposition: Let \( a \) be a real number. Prove that \( |a| = \max\{a, -a\} \).

Proof: Let \( a \) be a real number. Then there are three cases.

First suppose \( a > 0 \). Then by definition of absolute value \( |a| = a \) and by exercise 1.3.1 \( -a < 0 \). Because \( -a < 0 \) and \( 0 < a \) then by transitivity \( -a < a \) which by definition of maximum means that \( \max\{a, -a\} = a = |a| \).

Next suppose that \( a < 0 \) and thus by definition of absolute value \( |a| = -a \) Again by exercise 1.3.1 \( -a > 0 \). Because \( a < 0 \) and \( 0 < -a \) then by transitivity \( a < -a \) which by definition of maximum means that \( \max\{a, -a\} = -a = |a| \).

Lastly suppose that \( a = 0 \). Therefore \( |0| = 0 \) by definition of absolute value. Note that \( 0 \geq 0 \) and thus by definition of maximum \( \max\{0, 0\} = 0 = |0| \).

Moreover because \( a \) is a real number it must fall into one of these three cases by Axiom II then \( |a| = \max\{a, -a\} \). ☐