Prop.: Let $a$ and $b$ be real numbers. Then $|a + b| \leq |a| + |b|$.

**Proof**

Let $a$ and $b$ be real numbers.

**Case 1:** Let $a, b \geq 0$. Thus, $a, b \in \mathbb{R}^+$. By closure of the positive reals, $a + b \in \mathbb{R}^+$. By definition of absolute value, $|a + b| = a + b = |a| + |b|$.

**Case 2:** Let $a, b < 0$. Thus, $a, b \in \mathbb{R}^-$. By closure the negative reals, $a + b \in \mathbb{R}^-$. Using part 4 of Theorem 1.3.8, $|a + b| = |-(a + b)|$. Since $a + b \in \mathbb{R}^-$, $-(a + b) \in \mathbb{R}^+$. Thus, by definition of absolute value, $-(a + b) = -a - b$. By definition of absolute value, $-a - b = |a| + |b|$. Thus, $|a + b| = |a| + |b|$.

Without loss of generality, in the next two cases, we will consider $a \geq 0$ and $b < 0$.

**Case 3:** Let $a \geq 0$, $b < 0$, and $a \geq -b$. Thus, $a + b \geq 0$. So, $|a + b| = a + b$, as in case 1. Now consider $(a - b) - (a + b)$. We can simplify this down to get $-2b$. We know that $b$ is negative, so $-2b$ is positive. Thus, $(a - b) - (a + b) \in \mathbb{R}^+$. So, $a + b < a - b$. By definition of absolute value, $a - b = |a| + |b|$. Thus, $|a + b| < |a| + |b|$.

**Case 4:** Let $a \geq 0$, $b < 0$, and $a < -b$. Thus, $a + b < 0$. So, $|a + b| = -a - b$, as in case 2. Now consider $(a - b) - (-a - b)$. We can simplify this down to get $2a$. Since $a$ is positive or zero, $2a$ is positive or zero, so $(a - b) - (-a - b) \in \mathbb{R}^+ \cup \{0\}$. So, $-a - b \leq a - b$. By definition of absolute value, $a - b = |a| + |b|$. Thus, $|a + b| \leq |a| + |b|$.

Taking all of these cases together, we can see that $|a + b| \leq |a| + |b|$.