Problem 1.4.2

(b) Proposition: Let $x, y \in \mathbb{R}$. Suppose that for all positive real numbers $\varepsilon$, $x < y + \varepsilon$. Prove that $x \leq y$.

Proof (Contradiction)

Let $x, y \in \mathbb{R}$. Suppose that for all positive real numbers $\varepsilon$, $x < y + \varepsilon$ and suppose $x > y$. This means that $(x - y) \in \mathbb{R}^+$, so by exercise 1.3.1 (1) we know $-(x - y) = (y - x) \in \mathbb{R}^-$. Since $\varepsilon$ is any positive real number and $(x - y) \in \mathbb{R}^+$, let’s set $\varepsilon = (x - y)$.

Given that $x < y + \varepsilon$, we know $(y + \varepsilon) - x \in \mathbb{R}^+$. Using substitution and the properties of the additive inverse:

$$(y + \varepsilon) - x = (y + (x - y)) - x = (y - y) + (x - x) = 0 + 0 = 0.$$  

This is a contradiction to $(y + \varepsilon) - x \in \mathbb{R}^+$ because the order axiom tells us that a real number can’t be both positive and equal to zero. Hence, it must be the case that $x \leq y$. ■