P3.1.10

Proposition 2.2 → 3: If there exists \( a \in X \) and \( r > 0 \) such that \( S \subseteq B_r(a) \), then given any \( b \in X \), there exists \( r > 0 \) such that \( S \subseteq B_r(b) \).

**Proof**

Let \( a \in X \) and \( r > 0 \) such that \( S \subseteq B_r(a) \), and let \( b \) be an arbitrary element in \( X \). By the triangle inequality property of metrics, \( d(b, a) + d(a, x) \geq d(b, x) \) for all \( x \in S \). Since \( x \in S \subseteq B_r(a) \), \( d(a, x) < r \), and so \( d(b, a) + r > d(b, a) + d(a, x) \geq d(b, x) \). Thus, \( d(b, a) + r > d(b, x) \). Define \( r_1 = d(b, a) + r \). Then, \( d(b, x) < r_1 \) for all \( x \in S \), and so \( x \in B_{r_1}(b) \) for all \( x \in S \) by definition of an open ball. Thus, \( S \subseteq B_{r_1}(b) \). Since \( b \) was arbitrary, it follows that there exists an \( r > 0 \) for each \( b \in X \) such that \( S \subseteq B_r(b) \). ■