Proof. Assume that for every $a \in X$ there is a $r > 0$ such that $S \subseteq B_r(a)$ and that $S$ has no finite diameter. Then pick an arbitrary $s \in X$ and identify an $r \in \mathbb{R}^+$ such that $S \subseteq B_r(s)$. Now, we know that there exist $a, b \in S$ such that $d(a, b) > 2r$ since there is no greatest value for the diameter of $S$. Then, since $X$ is a metric space we know that $d(a, b) \leq d(a, s) + d(b, s)$ but $d(a, b) > 2r$ while $d(a, s)$ and $d(b, s)$ are both less than or equal to $r$. Thus we wind up with the expression $d(a, s) + d(b, s) \leq 2r < d(a, b)$. This is a contradiction as $d(a, b)$ cannot be both less than or equal to, and simultaneously greater than, $d(a, s) + d(b, s)$. \qed