Presentation 3.1.7(c)

Proposition: Let $X$ be an arbitrary metric space and let $S$ be an arbitrary subset of $X$. Show that if $U$ is an open subset of $X$ then $U \cap S$ is an open subset.

Proof: Let $X$ be an arbitrary metric space and let $S$ be an arbitrary subset of $X$. Suppose $U$ is an open subset of $X$. Then by the definition of an open subset, $U$ is the union of a collection of balls and thus $U = \bigcup_{a \in \lambda} B^X_r(a)$. Now consider $U \cap S = (\bigcup_{a \in \lambda} B^X_r(a)) \cap S$. Then by the definition of intersection $(\bigcup_{a \in \lambda} B^X_r(a)) \cap S = \bigcup_{a \in \lambda} (B^X_r(a) \cap S)$. Then using 3.1.7(b) since $B^X_r(a) \cap S = B^S_r(a)$ then $\bigcup_{a \in \lambda} (B^X_r(a) \cap S) = \bigcup_{a \in \lambda} B^S_r(a)$. Moreover, since $U \cap S$ equals a collection of open balls in $S$ then by definition $U \cap S$ is an open subset of $S$. ■