Let $X$ be an unbounded metric space. Show $X$ contains a sequence with no convergent subsequence.

Proof. Let $X$ be an unbounded metric space, and let $(a_n)$ be a sequence in $X$ with the following property; $d(a_1, a_i) + 1 < d(a_1, a_i)$. We know such a sequence exists because of the unboundedness of $X$, which allows us to get as far as we want from $a_1$. Now assume that $(a_n)$ had a convergent subsequence $(a_{n_i})$ that converges to $a$ and let $\epsilon = .5$. It follows then that there exists an $M \in \mathbb{N}$ such that for all $m > M$, $d(a_{nm}, a) < .5$. Since $m + 1 > m$, it follows that $d(a_{nm+1}, a) < .5$. From the triangle inequality, it follows that $d(a_{nm}, a_{nm+1}) \leq d(a_{nm+1}, a) + d(a_{nm}, a) < 0.5 + 0.5 = 1$. By the definition of the sequence, we also know that $d(a_1, a_{nm}) + 1 < d(a_1, a_{nm+1})$. We see now that $d(a_1, a_{nm+1}) \leq d(a_1, a_{nm}) + d(a_{nm}, a_{nm+1}) < d(a_1, a_{nm}) + 1$. We see that this contradicts the definition of our series, and thus is not possible. Thus our series has no convergent subseries. \qed