a) Prove that a monotonic sequence of real numbers converges if and only if it is bounded.

**Proof**

Let \((a_n)\) be a monotonic sequence of real numbers.

\((\Rightarrow)\) Suppose \((a_n)\) converges. By P3.3.4a, \((a_n)\) is bounded.

\((\Leftarrow)\) Suppose \((a_n)\) is bounded. Thus, \((a_n)\) has a least upper bound, \(y\), and greatest lower bound, \(x\), such that \(x \leq a_n \leq y\) for all \(n \in \mathbb{N}\).

Case 1: Suppose \((a_n)\) is increasing. By theorem 1.4.4, for all \(\epsilon > 0\) there exists an \(a_k \in (a_n)\) such that \(|a_k - y| < \epsilon\). Since \((a_n)\) is increasing, for all \(n > k\), \(a_n \geq a_k\). Since \(y\) is a least upper bound, \(y \geq a_n \geq a_k\). By Theorem 1.3.6, \(y - a_n \leq y - a_k\). Also, \(y - a_n \geq 0\) and \(y - a_k \geq 0\). Thus, \(a_n - y \leq 0\) and \(a_k - y \leq 0\). So, \(|a_n - y| \leq |a_k - y|\). Now, \(|a_n - y| \leq |a_k - y| < \epsilon\). Thus, \(|a_n - y| < \epsilon\), so \(a_n \to y\), and \((a_n)\) is convergent.

Case 2: Suppose \((a_n)\) is decreasing. By theorem 1.4.4, for all \(\epsilon > 0\) there exists an \(a_k \in (a_n)\) such that \(|a_k - x| < \epsilon\). Since \((a_n)\) is decreasing, for all \(n > k\), \(a_n \leq a_k\). Since \(x\) is the greatest lower bound, for all \(n \in \mathbb{N}\), \(a_n \geq x\). Thus, for all \(n > k\), \(x \leq a_n \leq a_k\). By theorem 1.3.6, \(a_n - x \leq a_k - x\). Also, \(a_k - x \geq 0\) and \(a_n - x \geq 0\). Thus, for all \(n > k\), \(|a_n - x| \leq |a_k - x| < \epsilon\). So, \(|a_n - x| < \epsilon\) for all \(n > k\), so \(a_n \to x\) and \((a_n)\) is convergent.

b) Thm 3.4.9: Every bounded sequence of real numbers has a convergent subsequence.

**Proof** Let \((a_n)\) be a bounded subsequence of real numbers. By P0.4.5b, \((a_n)\) has a monotonic subsequence, \((a_{n_i})\). Since \((a_n)\) is bounded, it has a least upper bound and a greatest lower bound. Thus, \((a_{n_i})\) must have a least upper bound and greatest lower bound. Thus, \((a_{n_i})\) is bounded. By part (a), \((a_{n_i})\) converges. Thus, \((a_n)\) has a convergent subsequence.