Theorem 1. Let $X$ be a metric space, and let $A \subseteq X$. Then the set of limit points of $A$ is a closed subset of $X$.

Proof. Let $S$ be the set of limit points of $A$, and let $x$ be a limit point of $S$. We need to show that $x \in S$.

By Theorem 3.5.1(3), if $r_1 > 0$, then $B_{r_1}(x)$ contains infinitely many points of $S$; say $s$ is one of them. (Note that this means that $s$ is a limit point of $A$.) Since $B_{r_1}(x)$ is open, there exists $r > 0$ such that $B_r(s) \subseteq B_{r_1}(x)$. Again by Theorem 3.5.1(3), the ball $B_r(s)$ contains infinitely many points of $A$ since $s$ is a limit point of $A$. Thus we also have $B_r(x)$ with infinitely many points of $A$, so $x$ is a limit point of $A$. Therefore, by the definition of $S$, $x \in S$. Since $x$ was an arbitrary limit point of $S$, $S$ is closed. □