P 4.3.4

**Proposition**: The composition of continuous functions is continuous.

**Proof**

Let $X$, $Y$, and $Z$ be metric spaces and $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Let $a$ be arbitrary in $X$ and let $r > 0$. Then, since $g$ is continuous, there exists some $s > 0$ such that if $d_Y(f(a), f(x)) < r$ then $d_Z(g(f(a)), g(f(x))) < r$. Next, since $f$ is continuous and $s > 0$, there exists some $t > 0$ such that if $d_X(a, x) < t$ then $d_Y(f(a), f(x)) < s$. We thus have by transitivity that for $r > 0$, there exists $t > 0$ such that if $d_X(a, x) < t$ then $d_Z(g(f(a)), g(f(x))) < r$. Since $r$ was arbitrary, this holds for all $r > 0$, and so $g \circ f$ is continuous at $a$ by definition. Then, because $a$ is arbitrary in $X$, this holds for all $a \in X$ and so $g \circ f$ is continuous. ■