Consider the function \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = x^2 \). Using only the definition of uniform continuity, prove that \( f \) is uniformly continuous on \([0,1]\) but not on \([0,\infty)\).

\[
\text{Proof on } [0,1]: \text{ Let } \epsilon > 0. \text{ Choose } \delta = \frac{\epsilon}{2}. \text{ Suppose } |a - b| < \delta \text{ and } a, b \in [0,1]. \text{ Thus, } |f(a) - f(b)| = |a^2 - b^2| = |a - b||a + b|. \text{ Since } a, b \in [0,1], |a + b| \leq 2. \text{ So, } |a - b||a + b| < 2\delta = \epsilon. \text{ Thus, } f \text{ is uniformly continuous on } [0,1]. \]

\[
\text{Proof on } [0,\infty): \text{ Suppose there exists some } \delta > 0 \text{ such that } |a - b| < \delta \text{ and } |f(a) - f(b)| < \epsilon \text{ when } a, b \in [0,\infty). \text{ Let } a = 6\epsilon \text{ and } b = 6\epsilon + \frac{\delta}{2}. \text{ Thus, } |a - b| = \frac{\delta}{2} \text{ and } |a + b| = 12\epsilon + \frac{\delta}{2} > 2\epsilon. \text{ Thus, } |f(a) - f(b)| > \epsilon. \]

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