Let $I \subset \mathbb{R}$ and $f : I \to \mathbb{R}$. We say that $f$ has the intermediate value property provided that for all $a, b \in I$ if $\gamma$ lies between $f(a), f(b)$, there is some $c \in (a, b)$ such that $f(c) = \gamma$. Prove that if $f : I \to \mathbb{R}$ has the intermediate value property, then it cannot have any jump discontinuities.

Assume $f : I \to \mathbb{R}$ such that $f$ has a jump discontinuity at $n \in I$. Then $\lim_{x \to n^{-}} f(x)$ exists, call it $L$, and $\lim_{x \to n^{+}} f(x)$ exists, call it $R$, for $\eta > 0$. Further, let $L \neq R$. Then we can assume $L < R$ without loss of generality. So let $\epsilon = \frac{1}{3} d(L, R)$, and by definition 5.2.1, there is some $\delta_1 > 0$ such that if $x \in (n - \delta_1, n)$, then $d(f(x), L) < \epsilon$ and there is some $\delta_2 > 0$ such that if $x \in (n, n - \delta_2)$, then $d(f(x), R) < \epsilon$. So choose $\delta = \min(\delta_1, \delta_2)$ such that $(n - \delta, n + \delta) \subset I$. Now observe that by P1.4.8d, there is an $\alpha_1 \in (L + \varepsilon, R - \varepsilon)$. But then by P1.4.8d again, there is some $\alpha_2 \in (\alpha_1, R - \varepsilon)$. Therefore, there exists $\alpha_1$ and $\alpha_2 \in (L + \varepsilon, R - \varepsilon)$ such that $\alpha_1 \neq \alpha_2$. If $f$ had the intermediate value property, then there would be $c, d \in (n - \delta, n + \delta)$ such that $f(c) = \alpha_1$ and $f(d) = \alpha_2$. But $\alpha_1$ and $\alpha_2$ are not in $(L, L + \varepsilon) \cup (R - \varepsilon, R)$. For this $\delta$, $c, d$ cannot be in $n - \frac{\delta}{2}, n + \frac{\delta}{2}$. Then it must be true that $c, d = n$. Then, if $f$ has the intermediate value property, $n$ maps to $\alpha_1$ and $\alpha_2$ and is not a function. So, if $f$ has a jump discontinuity, $f$ cannot have the intermediate value property.