Problem 6.1.1.3

**Theorem 1.** If \((a_n)\) is a Cauchy sequence then it must be bounded.

*Proof.* Let \((a_n)\) be a Cauchy sequence. Let \(\epsilon > 0\), and then there exists \(N \in \mathbb{N}\) such that when \(n, m > N\), \(d(a_n, a_m) < \epsilon\). Consider that the \(\sup\{ d(a_i, a_k) : i, k > N \} < \epsilon\) by our supposition. Now, let \(b = \max\{ d(a_i, a_k) : i, k \leq N \}\), then \(\sup\{ d(a_i, a_k) : i, k \leq N + 1 \} = b\). Finally it is necessary to consider the case where (wolg) \(i > N\) and \(k < N\). Then \(\sup\{ d(a_i, a_k) \} \leq \sup\{ d(a_i, a_{N+1}) \} + \sup\{ d(a_{N+1}, a_k) \} \leq b + \epsilon\), which is true by the triangle inequality. Since the set of distances between elements in the sequence has a finite supremum, then we know by Theorem 3.1.12 and Definition 3.1.13 that the Cauchy sequence is bounded. \(\Box\)