Complete the proof of Theorem 9.2.6.
c) Use Theorem 9.2.2 to prove that the product rule holds.

**Proof** Let \( f \) and \( g \) be differentiable at \( x \in K \). By Theorem 9.2.2,

\[
    f(y) = f'(x)(y - x) + f(x) + r(y)
\]

\[
    g(y) = g'(x)(y - x) + g(x) + e(y)
\]

such that \( \frac{r(y)}{y-x} \) and \( \frac{e(y)}{y-x} \) both approach zero as \( y \to x \). Now consider \( fg \) at an arbitrary \( y \in K \).

\[
    (fg)(y) = f(x)g(x)
\]

\[
    = [f'(x)(y - x) + f(x) + r(y)][g'(x)(y - x) + g(x) + e(y)]
\]

\[
    = f'(x)g'(x)(y - x)^2 + f'(x)g(x)(y - x) + f'(x)e(y)(y - x) + f(x)g'(x)(y - x)
\]

\[
    + f(x)g(x) + f(x)e(y) + r(y)g'(x)(y - x) + r(y)g(x) + r(y)e(y)
\]

\[
    = [f'(x)g(x) + f(x)g'(x)](y - x) + f(x)g(x) + f'(x)g'(x)(y - x)^2 + f'(x)e(y)(y - x)
\]

\[
    + f(x)e(y) + r(y)g'(x)(y - x) + r(y)g(x) + r(y)e(y)
\]

(1)

Let \( R(y) = f'(x)g'(x)(y - x)^2 + f'(x)e(y)(y - x) + f(x)e(y) + r(y)g'(x)(y - x) + r(y)g(x) + r(y)e(y) \).

Now we can see that \( \frac{R(y)}{y-x} = f'(x)g'(x)(y - x) + f'(x)e(y) + \frac{f(x)e(y)}{y-x} + r(y)g'(x) + \frac{r(y)g(x)}{y-x} + \frac{r(y)e(y)}{y-x} \).

Each of these terms goes to zero as \( y \to x \), so \( \frac{R(y)}{y-x} \to 0 \) as \( y \to x \). Thus, \( (fg)(x) \) is differentiable with derivative \( f'(x)g(x) + f(x)g'(x) \). ■