Problem D.2.1

(b) Proposition: \((a_n) = -1, 1, -1, 1, -1, 1, \ldots\) does not converge.

Proof.

Let \(L\) be a real number. We have two cases.

Case 1: Suppose \(L = -1\). Let \(\varepsilon = 1\). Then no matter how large we pick \(N \in \mathbb{N}\), there will always be an \(n > N\) such that \(a_n = 1\). In this case, \(d(a_n, L) = d(1, -1) = 2 > 1 = \varepsilon\). Thus, the sequence cannot converge to \(L\).

Case 2: Suppose \(L \neq -1\). Let \(\varepsilon = \left| -1 - L \right|\). Again, no matter how large we pick \(N \in \mathbb{N}\), there will always be an \(n > N\) such that \(a_n = -1\). In this case, \(d(a_n, L) = d(-1, L) = \left| -1 - L \right| > \varepsilon\). Thus, the sequence cannot converge to \(L\).

Since in both cases we see that \((a_n)\) cannot converge to \(L\), it follows that \((a_n) = -1, 1, -1, 1, \ldots\) does not converge.