Math 441: Real Analysis  
Sarah Larsen - Presentation Problems

Problem E.2.1

Proposition: Let $f : \mathbb{R} \to \mathbb{R}$ be a function. In each of the following cases, use Definition 4.2.1 to show that $\lim_{x \to a} f(x)$ exists.

(a) $f(x) = 2x + 1; a = 3$.
(b) $f(x) = x^3 - x; a = 0$.

Proofs

(a) Let $\epsilon > 0$. Choose $\delta = \epsilon/2$. Suppose $0 < |x - 3| < \delta$. Based on the fact that $f(a) = 2(3) + 1 = 7$, we would expect $\lim_{x \to 3} f(x) = 7$. So consider the possibility that $L = 7$. Then
\[
d(f(x), L) = |(2x + 1) - 7|
= |2x - 6|
= 2 \cdot |x - 3|
< 2 \cdot \delta
= 2 \cdot \frac{\epsilon}{2}
= \epsilon
\]
We have shown that $d(f(x), L) < \epsilon$ when $L = 7$. It follows by definition 4.2.1 that $\lim_{x \to 3} f(x) = 7$.

(c) Let $\epsilon > 0$. Choose $\delta = \min(1, \epsilon/4)$. Suppose $0 < |x| < \delta$. Since $\delta \leq 1$, we know $x \in (-1, 1)$ which implies $|x + 1| \leq 2$ and $|x - 1| \leq 2$. Based on the fact that $f(a) = 0^3 + 0 = 0$, we would expect $\lim_{x \to 0} f(x) = 0$. So consider the possibility that $L = 0$. Then
\[
d(f(x), L) = |x^3 - x|
= |x \cdot (x^2 - 1)|
= |x \cdot (x - 1) \cdot (x + 1)|
= |x| \cdot |x - 1| \cdot |x + 1|
< \delta \cdot 2 \cdot 2
= \frac{\epsilon}{4} \cdot 4
= \epsilon
\]
We have shown that $d(f(x), L) < \epsilon$ when $L = 0$. It follows by definition 4.2.1 that $\lim_{x \to 0} f(x) = 0$.  

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