4) Prove that there exists a negative number.

Solution

Let there exist a $x \in \mathbb{R}$ and $y \in \mathbb{R}$ such that $x < y$. By the definition of a relation, $y - x \in \mathbb{R}^+$. Suppose $a \in \mathbb{R}^+$ such that $a = y - x$. Then by exercise E 1.3.1 part 1 and since $a$ is positive, $-a$ is the complement of $\mathbb{R}^+ \cup \{0\}$ which is $\mathbb{R}^-$. Through substitution, $-a \in \mathbb{R}^-$ becomes $-(y - x) \in \mathbb{R}^-$. Therefore since $\mathbb{R}^-$ is a non-empty set containing $-(y - x)$, then a negative number must exist. $\star$