Problem 1.4.8 (c)

Let $x$ and $y$ be real numbers with $x < y$. Show that for all $x, y \in \mathbb{R}$, there exists a rational number $r$ satisfying $x < r < y$

Proof:

Let $x, y \in \mathbb{R}$ such that $x < y$, let $N \in \mathbb{Z}^+$, and let $n \in \mathbb{Z}$. From 1.4.8 part (b) we know that $\frac{n-1}{N} \leq x < \frac{n}{N}$. To show that there exists a rational number $r$ where $x < r < y$, we can show that $x < \frac{n}{N} < y$ since $\frac{n}{N}$ is a rational number. Since we already know that $x < \frac{n}{N}$, we just need to show that $\frac{n}{N} < y$. Let’s let $\epsilon = \frac{y-x}{2}$, then from theorem 1.4.7, there exists a positive integer $N$ such that $\frac{1}{N} < \epsilon$ or $\frac{1}{N} < \frac{y-x}{2}$. We know that $x < \frac{n}{N} = \frac{n-1}{N} + \frac{1}{N} < \frac{n-1}{N} + \epsilon < x + \frac{y-x}{2} = \frac{x+y}{2} < y$. Thus, $x < \frac{n}{N} < y$ and there exists a rational number in $(x, y)$. Q.E.D.