2.2.2

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Real Analysis 12:40

P 2.2.2

Let \((X, d)\) be a metric space. Prove \(\forall a, b, c \in X\)
\[|d(a, b) - d(b, c)| \leq d(a, c)\]

Let \((X, d)\) be a metric space and \(a, b, c\) be arbitrary elements in \(X\). First let us consider

the triangle inequality as it is applied to our distance function. We know it is true \(d(a, b) \leq d(a, c) + d(c, b)\).
Due to the symmetrical properties of our distance function, this statement

is equivalent to \(d(a, b) \leq d(a, c) + d(b, c)\). And we can subtract \(d(b, c)\) from each side to see

\(d(a, b) - d(b, c) \leq d(a, c)\).

Now let us consider another triangle inequality, namely \(d(b, c) \leq d(b, a) + d(a, c)\) which we

know to be true. Due to the symmetrical properties of our distance function, this statement

is equivalent to \(d(b, c) \leq d(a, b) + d(a, c)\). And we can subtract \(d(a, b)\) from each side to see

\(-d(a, b) + d(b, c) \leq d(a, c)\). Now we see \(|d(a, b) - d(b, c)| = d(a, b) - d(b, c)\) for

\(d(a, b) - d(b, c) \geq 0\), and \(|d(a, b) - d(b, c)| = -d(a, b) + d(b, c)\) for \(d(a, b) - d(b, c) < 0\).

So for both possible cases, \(|d(a, b) - d(b, c)| \leq d(a, c)\) for our arbitrary \(a, b, c \in X\). Thus,

\(\forall a, b, c \in X, |d(a, b) - d(b, c)| \leq d(a, c)\).

Q.E.D.