Problem 2.2.3

Let \((X, d)\) be a metric space. Use mathematical induction to prove the following generalization of the triangle inequality. Suppose that \(a_1, a_2, \ldots, a_k\) are in \(X\) and that \(k \geq 3\). Then
\[
d(a_1, a_k) \leq \sum_{i=1}^{k-1} d(a_i, a_{i+1}).
\]

**Proof.** Let \((X, d)\) be a metric space and suppose \(a_1, a_2, \ldots, a_k\) are in \(X\) and \(k \geq 3\). We shall proceed by induction.

**Base Case** Suppose \(k = 3\). Then \(d(a_1, a_k) = d(a_1, a_3)\) and \(\sum_{i=1}^{k-1} d(a_i, a_{i+1}) = \sum_{i=1}^{2} d(a_i, a_{i+1})\).

Note that \(\sum_{i=1}^{2} d(a_i, a_{i+1}) = d(a_1, a_{1+1}) + d(a_2, a_{2+1}) = d(a_1, a_2) + d(a_2, a_3)\). By the Triangle Inequality property of our metric space \(X\), \(d(a_1, a_3) \leq d(a_1, a_2) + d(a_2, a_3)\), and thus \(d(a_1, a_3) \leq \sum_{i=1}^{2} d(a_i, a_{i+1})\).

**Induction Step** Suppose \(k \geq 3\) and \(d(a_1, a_k) \leq \sum_{i=1}^{k-1} d(a_i, a_{i+1})\). Note that \(\sum_{i=1}^{k-1} d(a_i, a_{i+1}) = d(a_1, a_2) + d(a_2, a_3) + \ldots + d(a_{k-1}, a_k)\). Then
\[
d(a_1, a_k) + d(a_k, a_{k+1}) \leq \left(\sum_{i=1}^{k-1} d(a_i, a_{i+1})\right) + d(a_k, a_{k+1})
\]
\[
d(a_1, a_k) + d(a_k, a_{k+1}) \leq d(a_1, a_2) + d(a_2, a_3) + \ldots + d(a_{k-1}, a_k) + d(a_k, a_{k+1})
\]

By the Triangle Inequality property of our metric space \(X\),
\[
d(a_1, a_{k+1}) \leq d(a_1, a_k) + d(a_k, a_{k+1}).
\]
Then by transitivity,

\[ d(a_1, a_{k+1}) \leq d(a_1, a_k) + d(a_k, a_{k+1}) \leq d(a_1, a_2) + d(a_2, a_3) + \ldots + d(a_{k-1}, a_k) + d(a_k, a_{k+1}) \]

\[ d(a_1, a_{k+1}) \leq d(a_1, a_2) + d(a_2, a_3) + \ldots + d(a_{k-1}, a_k) + d(a_k, a_{k+1}) \]

\[ d(a_1, a_{k+1}) \leq \sum_{i=1}^{(k+1)-1} d(a_i, a_{i+1}) \]

By the Principle of Mathematical Induction, \( d(a_1, a_k) \leq \sum_{i=1}^{k-1} d(a_i, a_{i+1}). \) \( \square \)