Theorem 1. Let $x$ and $y$ be distinct points in a metric space $X$. Then there exists $r > 0$ such that $B_r(x)$ and $B_r(y)$ are disjoint.

Proof. Let $r = \frac{d(x,y)}{2}$. If $z \in B_r(x) \cap B_r(y)$, then $d(x,y) \leq d(x,z) + d(z,y) < r + r = d(x,y)$, a contradiction. Therefore $B_r(x) \cap B_r(y) = \emptyset$. 

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