Proving: A convergent sequence in a metric space has a unique limit.

**Solution**

By Contradiction

Let $a_n$ be a convergent sequence in metric space $X$ such that $a_n \to x \in X$; there exists $N \in \mathbb{N}$ if $n > N$ then $d(a_n, x) < \varepsilon$. Suppose another limit exists, $a_n \to y \in X$ such that $x \neq y$ and there exists $N \in \mathbb{N}$ if $n > N$ then $d(a_n, y) < \varepsilon$. Additionally, suppose that $\varepsilon = \frac{d(x,y)}{2}$. Applying the property of symmetry, $d(a_n, x) = d(x, a_n)$ so $d(x, a_n) < \varepsilon$. Next, we use the triangle inequality and $d(a_n, y) + d(x, a_n) \geq d(x, y)$. On the left side of the equation, transitivity can be applied (using $d(a_n, y) < \varepsilon$ and $d(a_n, x) < \varepsilon$) which results in $d(a_n, y) + d(x, a_n) < 2\varepsilon$. Then we can substitute this in and $2\varepsilon > d(x, y)$. Using the original definition of $\varepsilon$ and substitution, $2\varepsilon > 2\varepsilon$. This statement is untrue so $x$ and $y$ can’t both be the limit if $x \neq y$. Therefore, $a_n$ has a unique limit. ★