Let $X$ be a metric space. Let $(a_i)$ be a sequence in $X$ converging to $a$. Show that the set consisting of all the points in the range of the sequence $(a_i)$ together with the limit $a$ is a closed subset of $X$.

Proof. By definition of closed set, $\text{ran}(a_i) \cup a$ is a closed set if it contains all its limit points. Call $\text{ran}(a_i) \cup a, S$. We will proceed by contradiction. Suppose that there exists $b \in X$ such that $b \neq a$, but $b$ is also a limit point of $S$. Then by theorem 3.5.1 there exists a sequence $(b_i)$ of points in $S - \{b\}$ converging to $b$. Each $b_i = a_j$, for $j \in \mathbb{N}$. Construct a subsequence of $(b_i)$ as follows:

Let

\[
\begin{align*}
c_1 &= b_1 \text{ or the first term in } b_i \neq a \\
c_2 &= b_k \text{ for the least } k \text{ such that } j_k > j_1 \text{ and } b_k \neq a \\
c_n &= b_l \text{ for the least } l \text{ such that } j_l > j_{n-1} \text{ and } b_l \neq a
\end{align*}
\]

Note that $(c_i)$ must converge to $a$ though because it is a subsequence of $(a_i)$. Further $(c_i)$ must converge to $b$ because it is a subsequence of $(b_i)$. Therefore by the uniqueness of limits it must be that $a = b$. So it follows that there is only one limit point $a$ of the set $S$ and $a$ is included in the set, therefore the set is closed. \qed