Problem 4.4.1

Livingston, David

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Consider the function \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = x^2 \). Using only the definition of uniform continuity, prove that \( f \) is uniformly continuous on \([0, 1]\) but not on \([0, \infty)\).

Let \( \epsilon > 0 \). Let \( 0 < \delta < \frac{\epsilon}{2} \). Let \( a \) and \( b \) be two arbitrary elements of \([0, 1]\) such that \( d(a, b) < \delta \).

Then \( d(f(a), f(b)) = d(a^2, b^2) = |a^2 - b^2| \).

By theorem 1.3.8.5, \( |a^2 - b^2| = |a - b||a + b| \).

Observe that \( |a - b| = d(a, b) \) and that since \( a, b \in [0, 1] \), \( |a + b| \leq |1 + 1| = 2 \), so \( |a + b| \leq 2 \). Thus, \( |a - b||a + b| \leq 2d(a, b) \). Observe that \( 2d(a, b) < \epsilon \). By transitivity, \( d(f(a), f(b)) < \epsilon \). Since \( \epsilon \) was arbitrary, \( f \) is uniformly continuous on \([0, 1]\).

Let \( \epsilon > 0 \). Suppose that there exists some real number \( \delta > 0 \) s.t. if \( a, b \in [0, \infty) \) and \( d(a, b) < \delta \), then \( d(f(a), f(b)) < \epsilon \). Let \( a = \frac{\epsilon}{2} \) and let \( b = \frac{\epsilon}{2} + \delta \).

Since \( \epsilon > 0 \) and \( \delta > 0 \), by closure of positive numbers, \( a \) and \( b \) are in \([0, \infty)\). Observe that \( d(b, a) = |\frac{\epsilon}{2} + \delta - \frac{\epsilon}{2}| = |\frac{\delta}{2}| = \frac{\delta}{2} < \delta \).

Observe that \( b^2 = (\frac{\epsilon}{2} + \frac{\delta}{2})^2 = \frac{\epsilon^2}{4} + \epsilon + \frac{\delta^2}{4} \).

So \( d(f(a), f(b)) = |a^2 - b^2| = |\frac{\epsilon^2}{4} - \frac{\epsilon^2}{4} - \epsilon + \frac{\delta^2}{4}| = |\epsilon - \frac{\delta^2}{4}| = |\epsilon + \frac{\delta^2}{4}| = \epsilon + \frac{\delta^2}{4} > \epsilon \).

So \( d(f(a), f(b)) > \epsilon \). Therefore, \( f \) is not uniformly continuous on \([0, \infty)\).