Let $K$ be a bounded, nonempty subset of $\mathbb{R}$. Let $f : K \to \mathbb{R}$ be uniformly continuous. Prove $f$ is bounded, or that $f(K)$ is a bounded subset of $\mathbb{R}$.

Since $f$ is uniformly continuous, for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(a), f(b)) < \epsilon$ whenever $a, b \in K$ and $d_K(a, b) < \delta$. Let $\epsilon = 1$. So there exists a fixed $\delta$ such that $d_Y(f(a), f(b)) < 1$ whenever $a, b \in K$ and $d_K(a, b) < \delta$. Let $k$ be an arbitrary element of $K$. Consider the open ball of radius $\delta$ centered at $k$. Since $f$ is uniformly continuous, $f(B_\delta(k))$ has a radius of $\epsilon$, or 1. We can cover the entirety of the domain $K$ with balls and reach a finite size since $K$ is bounded, and the image of all these balls do not differ from the previous or next ball by length 1. So the range consists of the union of $n$ (finitely) many balls of radius 1. So $f(K)$, has a finite diameter (approximately $n$). Thus $f(K)$ is bounded and a subset of $\mathbb{R}$.

Q.E.D.