Prove that the continuous image of a compact set is compact.

**Solution**

Let $X$ be a compact metric space and $Y$ be a metric space. Also let $f : X \to Y$ be a continuous function. Suppose $S$ is compact in $X$ then $f(S) \in Y$. Additionally, suppose $f(S) \subseteq \bigcup_{\alpha \in \Lambda} U_{\alpha}$ which means we have an open cover for $f(S)$. Then by the definition of the inverse image, $f^{-1}(\bigcup_{\alpha \in \Lambda} U_{\alpha}) \in X$ and so $S \subseteq f^{-1}(\bigcup_{\alpha \in \Lambda} U_{\alpha})$. Next, we can pull the union from the inverse image such that $S \subseteq \bigcup_{\alpha \in \Lambda} f^{-1}(U_{\alpha})$ which can be expanded to $S \subseteq (f^{-1}U_{1} \cup f^{-1}U_{2} \cup \ldots \cup f^{-1}U_{n})$. By the definition of an image, $f(S) \subseteq (U_{1} \cup U_{2} \cup \ldots \cup U_{n})$. Thus we have a finite subcover of $f(S)$ and therefore $f(S)$ is compact. ⚫