Proposition: Let $X$ be a compact metric space, and let $Y$ be a metric space. If $f : X \to Y$ is a continuous function, then $f$ is uniformly continuous.

Proof. (Contrapositive) Let $X$ and $Y$ be as defined above, and suppose that $f : X \to Y$ is not a uniformly continuous function. To prove this statement via contraposition, it suffices to show that $f$ is not continuous, as we have supposed that $X$ is compact. By problem 4.4.3b, there exist some $\epsilon > 0$ and sequences $(x_n)$ and $(y_n)$ in $X$ such that $d_X(x_n, y_n) \to 0$, but for all $n \in \mathbb{N}$, $d_Y(f(x_n), f(y_n)) \geq \epsilon$. Because $X$ is compact, we can utilize Theorem 7.1.11. Thus, there exist subsequences $(x_{n_i})$ and $(y_{n_i})$ of $(x_n)$ and $(y_n)$, respectively, such that $x_{n_i} \to a$ and $y_{n_i} \to b$, for some $a, b \in X$.

We show that $a = b$. Suppose that $\xi$ is greater than 0. Because $d_X(x_n, y_n) \to 0$, there exists some $N_0$ such that for all $n > N_0$, $d_X(x_n, y_n) < \xi/3$. Also, by the definition of sequence convergence, we know that there exists some $N_1, N_2 \in \mathbb{N}$ such that for all $n_i > N_1$, $d_X(x_{n_i}, a) < \xi/3$, and for all $n_i > N_2$, $d_X(y_{n_i}, b) < \xi/3$. Let $N = \max\{N_0, N_1, N_2\}$, and let $n_i > N$. Therefore, by the triangle inequality,

$$d_X(a, b) \leq d_X(a, x_{n_i}) + d_X(x_{n_i}, y_{n_i}) + d_X(y_{n_i}, b) < \xi/3 + \xi/3 + \xi/3 = \xi,$$

so for all $\xi > 0$, $d_X(a, b) \leq \xi$, so by P1.4.2a, $d_X(a, b) = 0$, so $a = b$, by the positive definite property of the metric $d_X$.

Thus we have sequences $(x_{n_i})$ and $(y_{n_i})$, both converging to $a$ in $X$, but $d_Y(f(x_{n_i}), f(y_{n_i})) \geq \epsilon$, for all $n_i$ in the natural numbers. It follows that either $(f(x_{n_i}))$ or $(f(y_{n_i}))$ does not converge to $f(a)$, so without loss of generality, suppose that $(f(x_{n_i}))$ does not converge to $f(a)$. By negation of Theorem 4.3.3, $f$ is not continuous at $a$, so $f$ is not continuous. \qed