Problem 9.4.4

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(Corollary 9.4.7) Let $K$ be a $D$-domain, and let $f : K \to \mathbb{R}$ be a function. Let $I \subseteq K$ be an interval. Prove that $f$ is constant on $I$ if and only if $f'(x) = 0$ for all $x \in I$.

Suppose that $f$ is constant on $I$. Then for some $c \in \mathbb{R}$, if $a \in I$, then $f(a) = c$. Now let $x \in I$. Observe that $f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} = (x) = \lim_{y \to x} \frac{c - c}{y - x} = \lim_{y \to x} \frac{0}{y - x} = 0$. Therefore, for all $x \in I$, $f'(x) = 0$.

Now suppose that $f$ is not constant on $I$. Then for some distinct $a$ and $b$ in $I$, $f(a) \neq f(b)$. Without loss of generality, suppose that $a < b$. Then $[a, b] \subseteq I$. So by the mean value theorem, there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$. Since $f(b) \neq f(a)$, $\frac{f(b) - f(a)}{b - a} \neq 0$. Hence, for some $c \in I$, $f'(c) \neq 0$. By contrapositive, if $f'(x) = 0$ for all $x \in I$, then $f$ is constant.