Problem D.2.2(b)
February 25, 2013
(sorry this is so late)

Show, using Definition 3.3.1, that \( a_n = \left( \frac{1}{n}, \frac{1}{n^2} \right) \) is a convergent sequence.

Proof. Let \((a_n)\) be a sequence of real numbers defined \( a_n = \left( \frac{1}{n}, \frac{1}{n^2} \right) \). Let \( \varepsilon > 0 \) and let \( a = (0,0) \). Note that because \( \varepsilon > 0 \), then \( \frac{\varepsilon}{\sqrt{2}} > 0 \) as well. By Theorem 1.4.7, there exists \( N \in \mathbb{N} \) such that \( \frac{1}{N} < \frac{\varepsilon}{\sqrt{2}} \) (since it’s for all positive real numbers). Note also that \( \frac{1}{n^2} + \frac{1}{n^2} < \frac{1}{n^2} + \frac{1}{n^2} \), which will come up shortly. Thus, for all \( n > N \), it follows that

\[
d(a_n, a) = \sqrt{\left( \frac{1}{n} - 0 \right)^2 + \left( \frac{1}{n^2} - 0 \right)^2} = \sqrt{\frac{1}{n^2} + \frac{1}{n^4}} < \sqrt{\frac{1}{n^2} + \frac{1}{n^2}} = \sqrt{\frac{2}{n^2}} = \sqrt{2} \frac{1}{n}.
\]

(1)

And because (by Problem 1.3.9) \( \frac{\sqrt{2}}{n} < \frac{\varepsilon}{\sqrt{2}} < \varepsilon \), by transitivity \( d(a_n, a) < \varepsilon \) and by definition, \((a_n)\) converges to \((0,0)\). \( \square \)