Presentation Problem D.2.3b

D.2.3b: In $\mathbb{R}^3$: Prove whether $(\frac{1}{n}, \frac{1}{n^2}, \frac{1}{n^3})$ does or does not converge.

Proof. Let $\varepsilon > 0$. We claim that $(\frac{1}{n}, \frac{1}{n^2}, \frac{1}{n^3})$ converges to $(0,0,0)$. Let $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{\sqrt{3}}$ (and so $\frac{\sqrt{3}}{N} < \varepsilon$). By our claim,

$$d((\frac{1}{n}, \frac{1}{n^2}, \frac{1}{n^3}), (0,0,0)) = \sqrt{(\frac{1}{n} - 0)^2 + (\frac{1}{n^2} - 0)^2 + (\frac{1}{n^3} - 0)^2} = \sqrt{\frac{1}{n^2} + \frac{1}{n^4} + \frac{1}{n^6}}$$

We know that $\frac{1}{n^7} < \frac{1}{n^2}$ and $\frac{1}{n^6} < \frac{1}{n^2}$. So

$$\sqrt{\frac{1}{n^2} + \frac{1}{n^4} + \frac{1}{n^6}} < \sqrt{\frac{1}{n^2} + \frac{1}{n^4} + \frac{1}{n^6}} = \sqrt{\frac{3}{n^2}} = \frac{\sqrt{3}}{n}$$

If $n > N$, then

$$\frac{\sqrt{3}}{n} < \frac{\sqrt{3}}{N} < \varepsilon$$

Then by transitivity, $d((\frac{1}{n}, \frac{1}{n^2}, \frac{1}{n^3}), (0,0,0)) < \varepsilon$ for this $\varepsilon$, and since this $\varepsilon$ was arbitrary, then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > N$, then $d((\frac{1}{n}, \frac{1}{n^2}, \frac{1}{n^3}), (0,0,0)) < \varepsilon$. Therefore, $(\frac{1}{n}, \frac{1}{n^2}, \frac{1}{n^3}) \rightarrow (0,0,0)$.