1. Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. Rewrite Definition 4.2.1 in language that takes into account the fact that both the domain and the range of the function are \( \mathbb{R} \) under the usual metric. In each of the following cases, use your definition to show that \( \lim_{x \to a} f(x) \) exists.

(a) \( f(x) = 2x + 1; \ a = 3 \)

**Proof.** Let \( \varepsilon > 0 \), and let \( \delta = \frac{\varepsilon}{2} \). Suppose that \( 0 < |x - 3| < \delta \). Then

\[
d(f(x), f(a)) = |(2x + 1) - 7| = |2x - 6| = 2|x - 3| < 2\delta = 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.
\]

Therefore, by definition 4.2.1, \( \lim_{x \to 3} f(x) \) exists.

(c) \( f(x) = x^3 - x; \ a = 0 \)

**Proof.** Let \( \delta > 0 \) and note that \( x \in (-1, 1) \) because \( \delta \leq 1 \) (If you’re confused about this, I recommend reading page 320). Let \( \varepsilon > 0 \) and let \( \delta = \min(1, \frac{\varepsilon}{4}) \). Suppose that \( 0 < |x| < \delta \). Then

\[
d(f(x), f(a)) = |(x^3 - x) - 0| = |x^3 - x| = |x||x^2 - 1| = |x||x - 1||x + 1|
\]

So now, we want to make this distance as great as possible—sort of a worst case scenario—in order to really test if we can get a distance less than \( \varepsilon \). (Remember that \( |x| < \delta \)) So now let’s [sort of] substitute in the worst case scenario values of \( x \):

\[
|x||x - 1||x + 1| < \delta \cdot | -1 - 1 \cdot |1 + 1| = \delta \cdot 2 \cdot 2 = 4\delta \leq 4\left(\frac{\varepsilon}{4}\right) = \varepsilon
\]

Therefore, by definition 4.2.1, even in the worst conditions, \( \lim_{x \to 0} f(x) \) exists.

(e) \( f(x) = \frac{1}{x - 4}; \ a = 2 \)

**Proof.** Note that \( x \in (1, 3) \). Let \( \varepsilon > 0 \) and let \( \delta = \min(1, 2\varepsilon) \). Suppose that \( 0 < |x - 2| < \delta \). Then

\[
d(f(x), f(a)) = \left| \frac{1}{x - 4} - \frac{1}{-2} \right| = \left| \frac{1}{x - 4} + \frac{1}{2} \right| = \left| \frac{x - 4 + 2}{2(x - 4)} \right| = \frac{|x - 2|}{2(x - 4)}
\]

Now, recalling that \( |x - 2| < \delta \) and looking at the worst case scenario values of \( x \),

\[
\frac{|x - 2|}{2(x - 4)} < \frac{\delta}{2(3 - 4)} = \frac{\delta}{2} \leq \frac{2\varepsilon}{2} = \varepsilon
\]

Therefore, by definition 4.2.1, even in the worst possible scenario, \( \lim_{x \to 2} f(x) \) exists.