1. Certainly \( e_H \in G \), and \( e_H e_H = e_H \) since \( e_H \) is the identity of \( H \). By Exercise 1 of 7.2, \( e_H = e_G \).

3. I will apply the one-step subgroup test in each case.

(a) Since the identity is in every subgroup of \( G \), \( H \cap K \) is nonempty. Let \( a, b \in H \cap K \). Then \( a, b \in H \) and \( a, b \in K \), and \( ab^{-1} \in H \) and \( ab^{-1} \in K \) since both are subgroups of \( G \). Thus \( ab^{-1} \in H \cap K \), so \( H \cap K \leq G \).

(b) As before, \( \cap H_i \) is nonempty since it contains the group identity. If \( a, b \in \cap H_i \), then \( a, b \in H_i \) for each \( i \). Thus \( ab^{-1} \in H_i \) for each \( i \) since \( H_i \) is a subgroup. Therefore, \( ab^{-1} \in \cap H_i \), and \( \cap H_i \leq G \).

5. Since \( G_1 \leq G \) and \( H_1 \leq H \), \( G_1 \) and \( H_1 \) are nonempty, so \( G_1 \times H_1 \) is also nonempty. Let \( a, b \in G_1 \times H_1 \). Then \( a = (g_1, h_1), b = (g_2, h_2) \) for some \( g_1, g_2 \in G_1 \) and \( h_1, h_2 \in H_1 \). Now \( ab^{-1} = (g_1, h_1)(g_2^{-1}, h_2^{-1}) = (g_1g_2^{-1}, h_1h_2^{-1}) \in G_1 \times H_1 \) since \( G_1 \) and \( H_1 \) are groups.

7. \( T \neq \emptyset \) since \( e \in T \). Let \( a, b \in T \), and suppose that \( o(a) = m, o(b) = n \). Then \( (ab^{-1})^{mn} = a^{mn}(b^{-1})^{mn} \)
using the fact that \( G \) is abelian, which becomes \( (a^m)(b^n)^{-1} = e \). Therefore, \( T \leq G \).

10. Let \( b \in G \). We must show that \( ab = ba \). Consider \( b^{-1}ab \). Squaring this gives \( (b^{-1}ab)^2b^{-1}ab = e \). Thus \( b^{-1}ab = e \) or \( b^{-1}ab = a \) since \( a \) is the only element of order 2. In the first case, we get \( ab = ba \), so \( ab = ba \).

19. Suppose that \( n \in \mathbb{Z} \) is a generator of \( \mathbb{Z} \). Then there is an integer \( m \) such that \( mn = 1 \). Thus \( m, n \in \{ \pm 1 \} \).

In particular, \( n = \pm 1 \).

21. (a) \( (1, 1), 2(1, 1) = (2, 2) = (0, 2), 3(1, 1) = (3, 3) = (1, 0), 4(1, 1) = (0, 1), 5(1, 1) = (1, 2), 6(1, 1) = (0, 0) \). Thus \( (1, 1) \) generates all of \( \mathbb{Z}_2 \times \mathbb{Z}_3 \), so \( \mathbb{Z}_2 \times \mathbb{Z}_3 \) is cyclic.

(b) \( \mathbb{Z}_2 \times \mathbb{Z}_4 \) is generated by \( (1, 0) \) and \( (0, 1) \). However, it is not cyclic: if \( (a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_4 \), then \( 4(a, b) = (0, 0) \), but \( \mathbb{Z}_2 \times \mathbb{Z}_4 \) is cyclic.

25. Since \( e \in C(a), C(a) \neq \emptyset \). I will apply the two-step subgroup test. Let \( x, y \in C(a) \). Then \( (xy)a = x(ya) = x(ay) = (ax)y = a(xy) \), so \( xy \in C(a) \). Since \( ax = xa, x^{-1}a = ax^{-1} \), so \( x^{-1} \in C(a) \), as well. Therefore, \( C(a) \leq G \).

31. Since \( H \neq \emptyset \), \( x^{-1}Hx \neq \emptyset \). Let \( a, b \in x^{-1}Hx \). Then \( a = x^{-1}hx, b = x^{-1}kx \) for some \( h, k \in H \). Now \( ab^{-1} = (x^{-1}hx)(x^{-1}kx)^{-1} = (x^{-1}hx)(x^{-1}k^{-1}x) = x^{-1}hk^{-1}x \in x^{-1}Hx \), so \( x^{-1}Hx \leq G \).

43. Suppose that \( \frac{a}{b} \in \mathbb{Q} \) generates \( \mathbb{Q} \). Then there is a positive integer \( n \) such that \( n\frac{a}{b} = \frac{a}{2b} \). But this implies that \( 2abn = ab \), so \( 2n = 1 \), which is impossible. Therefore, \( \mathbb{Q} \) is not cyclic.