The Friendship Theorem: Graphs ♡ Matrices

John Caughman
Portland State University

Abstract: A colorfully named result in graph theory is known as the "Friendship Theorem," which can be stated roughly as follows:

Suppose that in a group of at least three people, each pair has precisely one common friend in the group. Then there must be someone who is everybody's friend.

Generally attributed to Erdös (1966), this result can be easily (and beautifully) proved using techniques from linear algebra. Interestingly enough, however, a combinatorial proof is not nearly so easy to come by. To be sure, a small number of combinatorial proofs have been found over the years; yet none of them come close to the beauty and simplicity of the algebraic proof.

The theorem (and its proof) have received no shortage of attention. In fact, the Friendship Theorem is listed among Paul and Jack Abad's "100 Greatest Theorems" – a surprising fact, given its simplicity. The algebraic proof, however, has managed to achieve even greater fame, being immortalized with the best of the best in Aigner and Ziegler's *Proofs from THE BOOK*. In this talk, we will try to appreciate what all the hype is about, discussing both algebraic and combinatorial methods of proof. We will discuss some connections this theorem has with other areas of mathematics, and, time permitting, we will consider several variations on this gem of algebraic graph theory.