Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$, or the limit doesn't exist, then the series $\sum_{n=1}^{\infty} a_n$ DIVERGES.

WARNING!! If $\lim_{n\to\infty} a_n = 0$, the test for divergence tells you NOTHING!

The Integral Test

If $f(x)$ is continuous, positive, and decreasing on $[1, \infty)$, and $a_n = f(n)$, then

(i) if $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$, is also convergent.

(ii) if $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$, is also divergent.

WARNING!! If $\int_1^{\infty} f(x)dx$ converges to a number, say for example 7, this tells us NOTHING about what the series $\sum a_n$ converges to.

Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with POSITIVE terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum a_n$ is divergent and $a_n \leq b_n$ for all n, then $\sum b_n$ is also divergent.

WARNING!! If $\sum b_n$ is divergent and $a_n \leq b_n$ this tells you nothing about $\sum a_n$.
Similarly, if $\sum a_n$ is convergent and $a_n \leq b_n$ this tells you nothing about $\sum b_n$.

Limit Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with POSITIVE terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$

where c is a finite number and $c > 0$, then either both series converge or they both diverge.

WARNING!! If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ the Limit Comparison test tells you NOTHING.

Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^n b_n = b_1 - b_2 + b_3 - b_4 + \ldots \qquad b_n > 0$$

satisfies: (i) $b_{n+1} \leq b_n$ for all n, and (ii) $\lim_{n\to\infty} b_n = 0$,

then the alternating series converges.

Alternating Series Estimation Theorem

If $\sum_{n=1}^{\infty} (-1)^n b_n$ satisfies the alternating series test, then the sum $s = \sum_{n=1}^{\infty} (-1)^n b_n$ is approximately equal to the partial sum $s_k = \sum_{n=1}^{k} (-1)^n b_n$,
and the error in this approximation is less than or equal to b_{k+1}.

Ratio Test

(i) If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L < 1$,
then the series is absolutely convergent, and hence convergent.

(ii) If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L > 1$,
then the series is divergent.

Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$,
then the series is absolutely convergent, and hence convergent.

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$,
then the series is divergent.

WARNING!! If the limits in the Ratio or Root test come out to be 1, this tells you NOTHING.

Important Examples

a) Geometric Series:

$$a + ar + ar^2 + ar^3 + ar^4 + \ldots = \sum_{n=1}^{\infty} a(r)^{n-1}$$

This geometric series converges if $|r| < 1$, and diverges if $|r| \geq 1$.
If $|r| < 1$, then the sum of the series is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$.

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b) p-series:

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

The p-series converges if $p > 1$, and diverges if $p \leq 1$.
This can be shown using the integral test.

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c) Harmonic series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n}$$

This is a p-series for $p = 1$ and it is divergent by the integral test.

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d) Alternating Harmonic series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

The alternating Harmonic series converges by the Alternating series test.
It is an example of a series which is convergent, but not absolutely convergent, i.e. the alternating harmonic series is conditionally convergent.