Discrete dynamical systems and Markov chains

A ***dynamical system*** models the movement of objects between locations or states over time. The vector summarizing the locations of the objects at a given time is called the ***state vector***, and the matrix expressing how the objects move between states is the ***transition matrix***.

If $\vec{x}_0$ represents the initial states of the system at time 0, and T is the transition matrix, then $\vec{x}_1 = T\vec{x}_0$ represents the states at time 1, $\vec{x}_2 = T\vec{x}_1 = T^2\vec{x}_0$ represents the states at time 2, etc.

A ***Markov chain*** is a dynamical system where the vectors $\vec{x_1}$, $\vec{x_2}$, ..., represent probabilities. In this context, each column of the transition matrix is a ***probability vector*** whose entries are all between 0 and 1, and the sum of the entries in each column is 1. A matrix whose columns are all probability vectors is called a ***stochastic matrix***.

A stochastic matrix P is ***regular*** if some positive power of P has all positive entries.

Theorem. If P is a regular stochastic matrix, then

- There is a unique vector $\vec{q}$ such that $P\vec{q} = \vec{q}$.
- For any initial probability vector $\vec{x}_0$, the sequence $\vec{x}_0$, $P\vec{x}_0$, $P^2\vec{x}_0$, ... converges to $\vec{q}$ as a limit.