Determinants

The ***determinant*** of an $n \times n$ matrix is a number $\det(A) = |A|$ associated with A. We can calculate $\det(A)$ in several different ways:

1. **(Diagonal arrows:)** For a 2×2 or 3×3 matrix, multiply down all diagonals, add the products along the right-slanting diagonals, and subtract the products along the left-slanting diagonals.
2. **(Elementary products:)** Multiply all permutation patterns of A (exactly one entry from each row and each column).
   Add the even permutations and subtract the odd permutations (even/odd minimum number of swaps to put the entries back in order).
3. **(Cofactor expansion:)**
   $\det(A) = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj}$ (expansion along the jth column)
   $\det(A) = a_{i1}C_{i1} + \cdots + a_{in}C_{in}$ (expansion along the ith row)
   where $C_{ij} = (-1)^{i+j}\det(A_{ij})$, where A_{ij} is obtained by deleting the ith row and jth column of A.

What are the determinants of these matrices?

1. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
3. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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1. $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$
2. $\begin{bmatrix} 3a & 3b & 3c \\ d & e & f \\ g & h & i \end{bmatrix}$
3. $\begin{bmatrix} a & b & c \\ d+5a & e+5b & f+5c \\ g & h & i \end{bmatrix}$
4. $\begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$

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Lemma. $\det(EA) = \det(E)\det(A)$, where E is an elementary matrix.

Theorem. $\det(AB) = \det(A)\det(B)$ for any matrices A and B.

Properties of Determinants

1. If A is an upper or lower triangular matrix (or a diagonal matrix) then $\det(A)$ is the product of its diagonal entries.
2. $\det\left(A^T\right) = \det(A)$.
3. A is invertible if and only if $\det(A) \neq 0$.
4. If A is invertible, then $\det(A^{-1}) = 1/\det(A)$.
5. If A has a row or column of zeros, two identical rows or columns, or two proportional rows or columns, then $\det(A) = 0$.
6. For an $n \times n$ matrix A and scalar k, $\det(kA) = k^n \det(A)$.
7. The determinant of a rotation in $\mathbb{R}^2$ or $\mathbb{R}^3$ is 1.
   The determinant of a reflection in $\mathbb{R}^2$ or $\mathbb{R}^3$ is -1.
8. If A is a 2×2 matrix, then $|\det(A)|$ is the area of the parallelogram defined by the columns of A.
9. If A is a 3×3 matrix, then $|\det(A)|$ is the volume of the parallelepiped defined by the columns of A.