Eigenthings

If A is a square matrix, $\vec{x}$ is a non-zero vector, and λ (lambda) is a scalar, and

$$A\vec{x} = \lambda\vec{x}$$

then λ is called an ***eigenvalue*** of A and $\vec{x}$ is called an ***eigenvector*** of A corresponding to λ.

$$\begin{aligned} A\vec{x} &= \lambda\vec{x} \\ A\vec{x} &= \lambda I_n\vec{x} \\ (A - \lambda I_n)\vec{x} &= \vec{0} \end{aligned}$$

Given an eigenvalue λ, the set of eigenvectors with eigenvalue λ is the set of solutions to $(A - \lambda I_n)\vec{x} = \vec{0}$, which is the null space of the matrix $A - \lambda I_n$, so it's a subspace of $\mathbb{R}^n$, called the ***eigenspace*** of λ.

Eigenstuff

We know $\text{null}(A - \lambda I_n) = \{\vec{0}\}$ if and only if $A - \lambda I_n$ is invertible.

Since we ignore $\vec{0}$ as an eigenvector, λ is an eigenvalue of A if and only if $A - \lambda I_n$ is not invertible, which is true if and only if $\det(A - \lambda I_n) = 0$.

Thinking of λ as a variable, this is is called the ***characteristic equation*** of A, and $\det(A - \lambda I_n)$ is called the ***characteristic polynomial*** of A.

Theorem

The number k is an eigenvalue of A if and only if k is a root of the polynomial $\det(A - \lambda I_n)$.

The number of times k is a root of $\det(A - \lambda I_n)$ is called the ***algebraic multiplicity*** of k as an eigenvalue of A.

Theorem

A is invertible if and only if 0 is not an eigenvalue of A.