Linear independence

• Recall that the vector \vec{w} is a *linear combination* of the vectors $\vec{v}_1, \ldots, \vec{v}_k$ if there exist scalars c_1, \ldots, c_k such that

$$\vec{w} = c_1 \vec{v}_1 + \ldots + c_k \vec{v}_k.$$

A set of vectors {v₁,..., v_k} is *linearly independent* if the only linear combination of the vectors that gives 0 is 0v₁ + ... + 0v_k. More formally, {v₁,..., v_k} is linearly independent if c₁ v₁ + ... + c_k v_k = 0 implies c₁ = c₂ = ··· = 0.

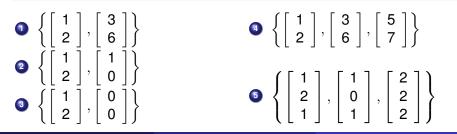
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Which sets of vectors are linearly independent?



- The columns of a matrix *A* are linearly independent if and only if $A\vec{x} = \vec{0}$ has exactly one solution, namely, $\vec{x} = \vec{0}$.
- A set consisting of a single vector is linearly independent if and only if the vector is not the zero vector.
- A set of two vectors is linearly independent if and only if the vectors are both not zero, and not multiples of each other.
- Any set of vectors {v
 ₁, v
 ₂, ..., v
 _p} in ℝⁿ is linearly dependent if p > n.
- Solution Any set of vectors $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ which contains the zero vector is linearly dependent.
- If $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ is linearly independent, then $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_{p-1}\}$ is also linearly independent.
- If $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_{p-1}\}$ is linearly dependent, then $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ is also linearly dependent.

Subspaces

A nonempty set W of vectors in \mathbb{R}^n is called a **subspace** of \mathbb{R}^n if

- If \vec{u} is in W then $c\vec{u}$ is in W for any real number c. (closed under scalar multiplication)
- 2 If \vec{u} is in W and \vec{v} is in W then $\vec{u} + \vec{v}$ is in W. (closed under vector addition)

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Which sets are subspaces?

- **1** The *x*-axis in \mathbb{R}^2 .
- 2 The line y = 2x + 3 in \mathbb{R}^2 .
- **3** The first quadrant in \mathbb{R}^2 .

- The solid ball $x^2 + y^2 + z^2 \le 1$ in \mathbb{R}^3 .
- The plane x + 2y + 3z = 0 in \mathbb{R}^3 .

The set of all linear combinations of the vectors $\vec{v}_1, \ldots, \vec{v}_k$ is called the **span** of $\vec{v}_1, \ldots, \vec{v}_k$ and is denoted span $\{\vec{v}_1, \ldots, \vec{v}_k\}$.

Theorems about subspaces

- A subset of \mathbb{R}^n is a subspace if and only if it's the span of some finite set of vectors $\vec{v_1}, \ldots, \vec{v_k}$.
- 2 $\{\vec{0}\}$ is a subspace of \mathbb{R}^n .
- 3 \mathbb{R}^n is a subspace of \mathbb{R}^n .
- ④ A subspace of ℝ² is either the origin, a line through the origin, or all of ℝ².
- S A subspace of \mathbb{R}^3 is either the origin, a line through the origin, a plane through the origin, or all of \mathbb{R}^3 .
- Solutions to the homogeneous matrix equation $A\vec{x} = \vec{0}$ is a subspace of \mathbb{R}^n .