Systems of linear equations

Why systems of linear equations are cool:

- Solutions to systems of linear equations can be thought of as solutions to matrix equations – solving the vector version of the equation $ax = b$, the equation $A\vec{x} = \vec{b}$ (Sections 3.1, 3.2, 3.3).

Why systems of linear equations are cool:

- Solutions to systems of linear equations can be thought of as solutions to matrix equations – solving the vector version of the equation $ax = b$, the equation $A\vec{x} = \vec{b}$ (Sections 3.1, 3.2, 3.3).
- Systems of linear equations describe linear objects in n-dimensional space: lines, planes, hyperplanes, etc. – in general called subspaces of $\mathbb{R}^n$ (Sections 3.4 and 3.5).

Why systems of linear equations are cool:

- Solutions to systems of linear equations can be thought of as solutions to matrix equations – solving the vector version of the equation $ax = b$, the equation $A\vec{x} = \vec{b}$ (Sections 3.1, 3.2, 3.3).
- Systems of linear equations describe linear objects in n-dimensional space: lines, planes, hyperplanes, etc. – in general called subspaces of $\mathbb{R}^n$ (Sections 3.4 and 3.5).
- Solutions to systems of linear equations can be thought of as inverse images of matrix functions: $A\vec{x} = \vec{b}$ asks for the set of all input vectors $\vec{x}$ that yield the output vector $\vec{b}$ (Sections 6.1–6.4).

- A ***system of linear equations*** is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

  This is called a system of m equations with n ***unknowns*.**

- A ***system of linear equations*** is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

  This is called a system of m equations with n ***unknowns*.**
- We write the system compactly using an ***augmented matrix***:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Elementary row operations on an augmented matrix:

- Multiply a row by a nonzero scalar.

Elementary row operations on an augmented matrix:

- Multiply a row by a nonzero scalar.
- Swap two rows.

Elementary row operations on an augmented matrix:

- Multiply a row by a nonzero scalar.
- Swap two rows.
- Add a multiple of one row to another.

- A matrix is in **reduced row echelon form** if it satisfies the following properties:
  1. Rows consisting of all zeros appear at the bottom of the matrix.

- A matrix is in **reduced row echelon form** if it satisfies the following properties:
  1. Rows consisting of all zeros appear at the bottom of the matrix.
  2. The first nonzero entry in each row with any nonzero entries is 1, called the *leading* 1 in that row.

- A matrix is in **reduced row echelon form** if it satisfies the following properties:
    1. Rows consisting of all zeros appear at the bottom of the matrix.
    2. The first nonzero entry in each row with any nonzero entries is 1, called the *leading* 1 in that row.
    3. If a column contains a leading 1, then all other entries in that column are zero.

- A matrix is in **reduced row echelon form** if it satisfies the following properties:
    1. Rows consisting of all zeros appear at the bottom of the matrix.
    2. The first nonzero entry in each row with any nonzero entries is 1, called the *leading* 1 in that row.
    3. If a column contains a leading 1, then all other entries in that column are zero.
    4. Each leading 1 is further to the right than any above it.

- A matrix is in **reduced row echelon form** if it satisfies the following properties:
  1. Rows consisting of all zeros appear at the bottom of the matrix.
  2. The first nonzero entry in each row with any nonzero entries is 1, called the *leading* 1 in that row.
  3. If a column contains a leading 1, then all other entries in that column are zero.
  4. Each leading 1 is further to the right than any above it.
- The location in the matrix corresponding to a leading 1 is called a ***pivot position***, and the column it's in is a ***pivot column***.
- The variables corresponding to the leading 1's are called ***leading variables***, and the others are ***free variables***.

- A matrix is in **reduced row echelon form** if it satisfies the following properties:
  1. Rows consisting of all zeros appear at the bottom of the matrix.
  2. The first nonzero entry in each row with any nonzero entries is 1, called the *leading* 1 in that row.
  3. If a column contains a leading 1, then all other entries in that column are zero.
  4. Each leading 1 is further to the right than any above it.
- The location in the matrix corresponding to a leading 1 is called a ***pivot position***, and the column it's in is a ***pivot column***.
- The variables corresponding to the leading 1's are called ***leading variables***, and the others are ***free variables***.
- If we perform a series of row operations to get matrix A into the matrix U in reduced row echelon form, then we say U is the ***reduced row echelon form*** of A.

- A matrix is in **reduced row echelon form** if it satisfies the following properties:
  1. Rows consisting of all zeros appear at the bottom of the matrix.
  2. The first nonzero entry in each row with any nonzero entries is 1, called the *leading* 1 in that row.
  3. If a column contains a leading 1, then all other entries in that column are zero.
  4. Each leading 1 is further to the right than any above it.
- The location in the matrix corresponding to a leading 1 is called a ***pivot position***, and the column it's in is a ***pivot column***.
- The variables corresponding to the leading 1's are called ***leading variables***, and the others are ***free variables***.
- If we perform a series of row operations to get matrix A into the matrix U in reduced row echelon form, then we say U is the ***reduced row echelon form*** of A.

Theorem

Every matrix has exactly one reduced row echelon form.

- A linear system is ***homogenous*** if all the *b*'s are 0.

- A linear system is ***homogenous*** if all the b's are 0.
- A linear system is ***consistent*** if it has at least one solution.

- A linear system is ***homogenous*** if all the b's are 0.
- A linear system is ***consistent*** if it has at least one solution.

Question: How many solutions can a linear system have?

- A linear system is ***homogenous*** if all the b's are 0.
- A linear system is ***consistent*** if it has at least one solution.

Question: How many solutions can a linear system have?

Theorem

Every linear system has either 0, 1, or ∞ solutions.

- A linear system is ***homogenous*** if all the b's are 0.
- A linear system is ***consistent*** if it has at least one solution.

Question: How many solutions can a linear system have?

Theorem

Every linear system has either 0, 1, or ∞ solutions.

The ***rank*** of a matrix A is the number of leading 1's / pivot columns / leading variables in its reduced row echelon form.

- A linear system is ***homogenous*** if all the b's are 0.
- A linear system is ***consistent*** if it has at least one solution.

Question: How many solutions can a linear system have?

Theorem

Every linear system has either 0, 1, or ∞ solutions.

The ***rank*** of a matrix A is the number of leading 1's / pivot columns / leading variables in its reduced row echelon form.
Suppose a linear system has m equations, n unknowns, and rank r.
(So the non-augmented matrix has m rows and n columns.)
How many solutions could the system have if:

1. $r = m$?
2. $r = n$?
3. $r < n$?
4. $r = m = n$?
5. $m < n$?

- A linear system is ***homogenous*** if all the b's are 0.
- A linear system is ***consistent*** if it has at least one solution.

Question: How many solutions can a linear system have?

Theorem

Every linear system has either 0, 1, or ∞ solutions.

The ***rank*** of a matrix A is the number of leading 1's / pivot columns / leading variables in its reduced row echelon form.
Suppose a linear system has m equations, n unknowns, and rank r.
(So the non-augmented matrix has m rows and n columns.)
How many solutions could the system have if:

1. $r = m$?
2. $r = n$?
3. $r < n$?
4. $r = m = n$?
5. $m < n$?
6. What if the linear system is homogeneous?