Vector Spaces

A (***real***) ***vector space*** V is a set of things called ***vectors*** together with two operations called ***addition*** and ***scalar multiplication*** satisfying the following properties:

1. If $\vec{u}$ and $\vec{v}$ are in V, then $\vec{u}+\vec{v}$ is in V. (closed under addition)
2. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ (addition is commutative)
3. $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$ (addition is associative)
4. There is a vector called $\vec{0}$ such that $\vec{0}+\vec{u}=\vec{u}$. (additive identity)
5. For each $\vec{u}$ there is a vector $-\vec{u}$ such that $\vec{u}+(-\vec{u})=\vec{0}$. (inverses)
6. If $\vec{u}$ is in V, then $c\vec{u}$ is in V. (closed under scalar multiplication)
7. $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$. (distributive law #1)
8. $(c+d)\vec{u}=c\vec{u}+d\vec{u}$. (distributive law #2)
9. $c(d\vec{u})=(cd)\vec{u}$ (scalar multiplication is associative)
10. $1\vec{u}=\vec{u}$.

Since linear combinations work in vector spaces, all linear algebra works in vector spaces!

Are these all vector spaces?

1. 2×2 matrices, with regular addition and scalar multiplication.
2. Complex numbers, with regular addition and (real) scalar multiplication.
3. The set of polynomials, with regular addition and scalar multiplication.
4. Polynomials of degree at most 2, with regular addition and scalar multiplication.

If they are, what is the (real) dimension of these vector spaces?

Vector space linear transformations

Are these linear transformations?

1. The function $T(a+bi) = a - bi$ from $\mathbb{C}$ to $\mathbb{C}$.
2. The function $T(\vec{v}) = ||\vec{v}||$ from $\mathbb{R}^2$ to $\mathbb{R}$.
3. The function $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a, b, c, d)$ from 2×2 matrices to $\mathbb{R}^4$.
4. The function $T(f) = f'$ from the set of polynomials of degree at most 2 to itself.
5. The function $T(A) = \mathrm{rref}(A)$ from 2×2 matrices to 2×2 matrices.
6. The function $T(p) = p(0)$ from the set of polynomials of degree at most 2 to $\mathbb{R}$.

For those that are, what is their range, kernel, rank, and nullity? What is their associated matrix?

Suppose V and W are vector spaces. An ***isomorphism*** is a bijective linear transformation $T : V \to W$.

V and W are ***isomorphic*** if there is an isomorphism between them.

Theorem

Every n-dimensional (real) vector space is isomorphic to $\mathbb{R}^n$.